WALLPAPER GROUPS AND THE MAGIC THEOREM

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ABSTRACT. The motivations of studying symmetry groups arise from understanding repetitive patterns in visual arts and architecture. There were significantly many discoveries about symmetry in the 19th century, one of which is the theorem of 17 plane symmetries. This paper discusses a special type of symmetry group —wallpaper groups. We will study wallpaper symmetries, prove the 17 plane symmetries with Conway's magic theorem and meanwhile bring in the application of symmetry in constructing wallpaper patterns. The study of symmetry is a core part of group theory, and it also talks with geometry, linear algebra and even architectural arts. In particular, the involvement of wallpaper patterns makes this topic appealing to a wide variety of readers; but the underlying mathematical theory will certainly spark the interest of students in mathematics. Many mathematical research fields involve symmetry groups: Lie group, combinatorial graph, molecular symmetry, quantum mechanics, etc.

1. SYMMETRY GROUPS

Symmetry in our mathematical studies is referred to as isometry. It can be in all dimensions, but for the sake of visualization, we are most interested in isometry of two dimensions.

Definition 1 (Isometry)**.** An *isometry* of $n-$dimensional space $\mathbb{R}^n$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ that preserves distance.

Whenever we talk about symmetry groups, we should understand that the set of all isometries defined as the above is indeed a group. Let us show that.

Theorem 2. *The set of all isometries in $F \subset \mathbb{R}^n$ is a group under function composition.*

Proof. Let G be the set of all isometries in $\mathbb{R}^n$ that map F onto itself.

(1) *G is closed under the function composition.*
For any $T_1, T_2 \in G$ and any $x, y \in F$, $\|T_2 \circ T_1(x) - T_2 \circ T_1(x)\| = \|T_1(x) - T_1(y)\| = \|x - y\|$. So the composition of two symmetry functions are still in G.
(2) *Identity*.
It is clear that $T(x) = x; F \to F$ is the identity function.
(3) *Associativity*
We can take it for granted that function composition is always associative.
(4) *Inverse*.
By definition of G, $\forall\, T \in G$, T is onto. Also notice that if $T(x) = T(y)$ for some $x, y \in F$, $0 = \|T(x) - T(y)\| = \|x - y\|$, which implies $x = y$. That is to say, T is one-to-one. Now, T is bijective in F, so its inverse T^{-1} exists. Moreover, $\forall\, y_1, y_2 \in F$, $\exists\, x_1, x_2 \in F$ such

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that $T(x_1) = y_1$ and $T(x_2) = y_2$, so $\|T^{-1}(y_1) - T^{-1}(y_2)\| = \|T^{-1} \circ T(x_1) - T^{-1} \circ T(x_2)\| = \|x_1 - x_2\| = \|y_1 - y_2\|$, which verified $T^{-1} \in G$.

In conclusion, the set G defined above is a group. $\square$

This leads to the following definition:

Definition 3 (Symmetry groups in $\mathbb{R}^n$)**.** Let $F \subset \mathbb{R}^n$, then the symmetry group of F is the set of all isometries of $\mathbb{R}^n$ that map F onto itself with the group operation function composition.

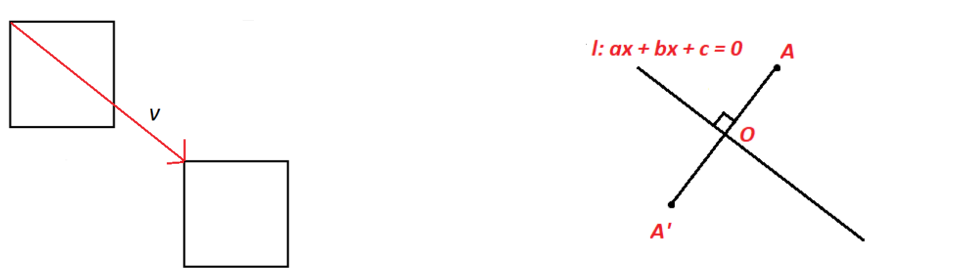
We have already seen some simple symmetry groups in class, such as the dihedral group D_n, which describes the symmetry groups of a regular polygon in two-dimensional space. We examined their properties by Cayley's table. And now with an introduction to symmetry groups in general, we are able to claim the following.

1. **Symmetry on a plane.** The symmetry groups on a plane $\mathbb{R}^2$ are easier to grasp than higher dimensions. It has been well known that rotation, reflection, translation and glide-reflection are the four classes of isometries in R^2. It will be interesting to study those four operations pointwisely by algebraic functions, so we will do that now (p 3-5, [3]).

Translation. Translation is an operation that has no fixed points and that moves every point towards a certain direction with a certain distance (see Fig 1a). Let $T_v(u) : \mathbb{R}^2 \to \mathbb{R}^2$ denote the function of translation, where $u \in \mathbb{R}^2$ and $v \in \mathbb{R}^2$ be the translation vector. So $T_v(u) = u + v$, which is analogous to vector addition in $\mathbb{R}^2$. A translation is said to be *nontrivial* if $v \neq 0$, ie, not the identity map.

we can check that such translation is *an isometry*. Given $u_1, u_2 \in \mathbb{R}^2$ and a translation function $T_v(u)$, $\|T_v(u_1) - T_v(u_2)\| = \|(u_1 + v) - (u_2 + v)\| = \|u_1 - u_2\|$. So translation is distance-preserving.

Also, the set of all translations under function composition is an *infinite* subgroup of the symmetry group on $\mathbb{R}^2$. The inverse of T_v is T_{-v}; it is closed since $T_{v_2} \circ T_{v_1} = T_{v_1+v_2}$. Moreover, the group is infinite because $T_v^n \neq T_0$ for any $n \geq 1$.



(a) translation

(b) reflection

Figure 1. Translation and reflection

Reflection. A reflection fixes all the points on a mirror line, and maps all the other points across that line (see Fig 1b). A reflection has an order of 2, so it is its own inverse. Let's call the reflection function $L_l(u) : \mathbb{R}^2 \to \mathbb{R}^2$, where $l \in \mathbb{R}^2$ is the mirror line.

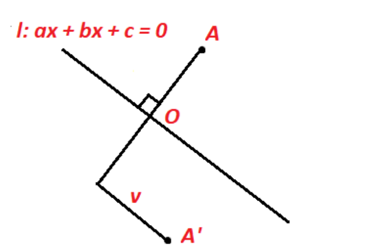
Suppose the standard form l is $ax + by + c = 0$, where $(x,\ y) \in \mathbb{R}^2$, $a, b, c \in \mathbb{R}$ and $a \neq 0$ or $b \neq 0$. Suppose $A : (x_0, y_0)$ is any point, then we try to find the symmetric point A'. Notice that the line $AA' : bx - ay + ay_0 - bx_0 = 0$ is a perpendicular bisector to line l, so we find the intersecting point $O : (\frac{b^2 x_0}{a^2+b^2} - \frac{ab\, y_0}{a^2+b^2} - \frac{ac}{a^2+b^2}, -\frac{ab\, x_0}{a^2+b^2} + \frac{a^2 y_0}{a^2+b^2} - \frac{bc}{a^2+b^2})$. Then, we compute $A' : (\frac{(b^2-a^2)\, x_0}{a^2+b^2} - \frac{2ab\, y_0}{a^2+b^2} - \frac{2ac}{a^2+b^2}, -\frac{2ab\, x_0}{a^2+b^2} + \frac{(a^2-b^2)\, y_0}{a^2+b^2} - \frac{2bc}{a^2+b^2})$. Now we see that $L_l(u) = \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2} \\ \frac{-2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \end{pmatrix} u + \begin{pmatrix} \frac{-2ac}{a^2+b^2} \\ \frac{-2bc}{a^2+b^2} \end{pmatrix}$ for all $u \in \mathbb{R}^2$.

First, from the algebraic equation, it shows that a reflection is a linear transformation. Besides, $\|L_l(u_1) - L_l(u_2)\| = \left\| \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2} \\ \frac{-2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \end{pmatrix} (u_1 - u_2) \right\| = \|u_1 - u_2\|$ implies a reflection $L_l(u)$ is an isometry.

Rotation. Rotation describes a movement of a line with a fixed point as the pivot (see Fig 2a). The angle of rotation is fixed in a way that a finite number of repetition on the rotation gives 360°, which is the identity. Algebraic equations for a rotation $r_\theta(u)$ of angle θ on the vector u is given by $r_\theta(u) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} u + \vec{C}$, where the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is called a rotation matrix in $\mathbb{R}^2$ and $\vec{C}$ is determined by the rotation center. Clearly, a rotation is a linear transformation.

Let us inspect the isometry property of a rotation. Given two points u_1 and u_2, then $\|r_\theta(u_1) - r_\theta(u_2)\| = \left\| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} (u_1 - u_2) \right\| = \|u_1 - u_2\|$, so a rotation preserves distance of two points. Clearly composition of two rotations is also a rotation. It is now verified that a rotation is an isometry.

(a) rotation



(b) glide reflection

Figure 2. Rotation and glide reflection

Glide reflection. The most obvious way to see a glide reflection is the composition of translation and reflection (see Fig 2b). So its algebraic expression can be written as $G(u) = L_l(u) + v = \begin{pmatrix} \frac{b^2 - a^2}{a^2 + b^2} & \frac{-2ab}{a^2 + b^2} \\ \frac{-2ab}{a^2 + b^2} & \frac{a^2 - b^2}{a^2 + b^2} \end{pmatrix} u + \begin{pmatrix} \frac{-2ac}{a^2 + b^2} \\ \frac{-2bc}{a^2 + b^2} \end{pmatrix} + v$. If the the glide reflection is not a reflection, we say that the glide reflection is nontrivial.

It is not hard to see that a glide reflection is an isometry because it is a composition of two isometries, translation and reflection.

2. **More properties on the plane symmetry operations.** We are also interested in other properties of the four operations, such as *orientation-preserving* , as we will show by observation. (p 5-7, [3]; chapter 3, [5])

(a) Orientation preserved

(b) orientation reserved

Figure 3. A geometric view of operations

As seen in Fig 3a and Fig 3b, there are two main types of operations, orientation preserving and non-orientation preserving. In Fig 3a if we trace the boundary counterclockwisely, we will have ACB and $A'C'B'$, so we say that the orientation is preserved; on the contrary, in Fig 3b, the boundaries are ACB and $A'B'C'$, so the orientation is reversed. Among our four operations, translation and rotation preserve the orientation, whereas reflection and glide reflection reverse the orientation.

Now, before we introduce the composition those operations, it helps to learn to recognize an operation. Given two images, before and after some operation, we should be able to deduce the operation that was done. Here are the main steps:

(1) We begin with examining whether the operation is a *translation*. Firstly, the two images must have have the same orientation; besides, find two correspondent points respectively for the two images, such as A and A', B and B' in Fig 3a and 3b, connect each two correspondent ones, and if the two lines are parallel then the operation is a translation.

(2) If it is not a translation, but the orientation is still preserved, then it is a *rotation*. It can be tricky to find the center of rotation, and here is how we can do it. Let the two triangles in Fig 3a be our example for illustration. We connect two pair of correspondent points, in this case A and A', B and B', and draw a bisector line l for line AA' and m for line BB'. Next, we find the intersecting point O of lines l and m, and that will be the center of rotation. The angle of rotation is given by $\angle AOA'$. The proof lies on the congruence of the triangles

$\triangle AOB$ and $\triangle A'OB'$ by the SSS law.

(3) Now, if the orientation has been changed, we can still connect two pairs of symmetric points, and if the two lines are parallel the operation is a *reflection*. The mirror line is the normal bisector line of those two lines, which is wv in Fig 1b.

(4) Lastly, for another case of reverse orientation (see Fig 3b, if the lines AA' and CC' connecting two pair of symmetric points intersect at some point, then the operation will be a *glide reflection*. And the mirror line can be found by connecting the midpoints of those two lines AA' and CC'.

Now we are able to tell the composition of any two of the four operations. The geometric intuition here is straightforward enough to figure the composition of two operation, so a rigorous proof is omitted. The following table (Table 1) presents the relations. The m th row and the n th column means $m \circ n$.

Operations	translation	reflection	rotation	glide reflection
translation	translation	glide reflection	rotation	glide reflection
reflection	glide reflection	rotation	glide reflection	translation
rotation	rotation	glide reflection	rotation	glide reflection
glide reflection	glide reflection	translation	glide reflection	translation

Table 1. Compositions of isometries

2. Wallpaper patterns and terminology

Though wallpaper patterns are not new to most of us, we will now look at them from a mathematical point of view. we will first introduce some notations, terminologies and a key theorem in plane symmetry, and then some representatives of different symmetry groups.

1. **Wallpaper groups.** Wallpaper groups, also called crystallographic group, are infinite discrete plane symmetry groups. They have repetitive patterns filling the whole plane $\mathbb{R}^2$, and the patterns are invariant under compositions of two linearly independent translations. A *lattice unit* is a parallelogram whose vertices are a particular point in the pattern, and it can generate the whole plane by translations in two directions of its sides; a lattice unit is not unique as we can choose a different point as vertices of the parallelogram. See the rectangular frames in Fig 4a and Fig 4b. In most cases, there are also other isometries on the wallpaper such as rotations, reflections and glide reflections. Necessarily, any isometry will map a lattice onto itself (p 441, [4]). It is straightforward to see that the group of all such translations that maintain the pattern are a subgroup of the wallpaper group (p 452-453, [1]). The two examples below are created on *Wallpaper Symmetry*[1]. In order to describe patterns of wallpaper groups, let us introduce the notations adopted by John H. Conway and William Thurston (Chapter 1 and 2, [2]).

[1]http://math.hws.edu/eck/js/symmetry/wallpaper.html

(a) p4, 442 **(b)** pmm, *2222

Figure 4. Two wallpaper patterns

Theorem 4. *There are exactly* 17 *wallpaper groups in* $\mathbb{R}^2$.

We will provide a sketch of the proof here, and the complete proof of it can be found in [6], which is accessible for most students familiar with group theory.

We introduce an operation $\forall x \in \mathbb{R}^2, (v, \phi)(x) = v + \phi x$ with $(v, \tau) = v + x$, where $v \in \mathbb{R}^2$ and ϕ is a linear transformation, and one can check that this operation concludes all the symmetries in $\mathbb{R}^2$, and it is operation-preserving. Then, we have the group of translation as $T = \{t \in \mathbb{R}^2 : (t, \tau) \in G\}$, the group of rotation or reflection $H = \{\phi : (v, \phi) \in G \text{ for some } v \in \mathbb{R}^2\}$. We now can define the group of symmetries as $G = T \cup H$, where $T = \{n_1 t_1 + n_2 t_2 : n_1, n_2 \in \mathbb{Z}, t_1, t_2$ independent$\}$, and H is finite. We split G into three cases: G has no reflection, one reflection or more than one reflection.

Case one (no reflection): For any two groups G and G' with the same subgroup H as the group of rotations, we can construct an isomorphism to show equivalence of G and G', which implies distinct rotation subgroups generate distinct symmetry groups G. There are 5 different rotation groups since order of a rotation group can be $1, 2, 3, 4$ or 6. Hence, there are 5 symmetry groups with no reflection.

Case two (one reflection): It is shown there are three possible situations, and again we construct isomorphisms to the uniqueness of those three cases. Then there are three symmetry groups with one single reflection.

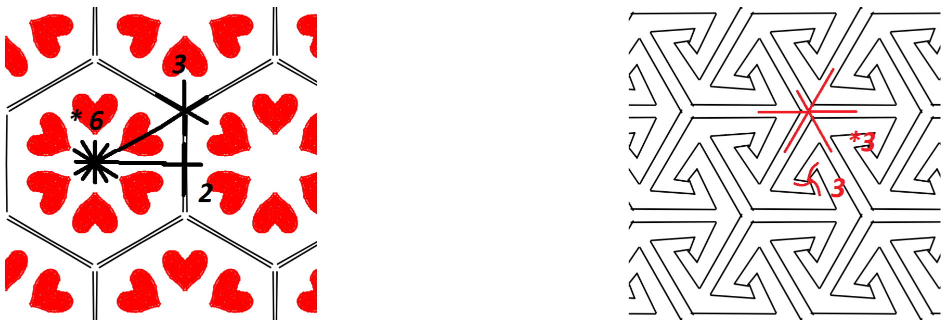
Case three (more than one reflection): In this case, the subgroup H of a symmetry group G contains two reflections ρ, σ, and two reflections together make a rotation. Denote the order q of rotation $\rho\sigma$, then cases are $q = 2$, which has 4 classes of groups, $q = 3$ with 2 classes, $q = 4$ with 2 classes and $q = 6$ with one class. Similar arguments from the previous two cases are applied here, and there are 9 distinct symmetry groups in this case.

Adding up all the possibilities in three cases gives us $5 + 3 + 9 = 17$ classes of symmetry groups. In the next section, we will also apply Conway's magic theorem to prove this theorem cleverly. To do that, we need to know about some signatures assigned to different groups.

2. **Orbifold signatures of wallpaper patterns.** The four operations reflection, translation, rotation and glide reflection have their own signatures, as described in Chapter 1 and 2 in [2]. Mirror

lines (kaleidoscope), gyrations, miracles and wonders are used to describe reflection, rotation, glide reflection and translation, respectively.

Kaleidoscopes (mirror lines). When we describe kaleidoscopes, we find out all the points where the mirror lines intersect. $*$ is used to denote existence of mirror lines, then followed by number of mirror lines at each intersecting point. If you see the example 5a, we have three different intersecting points from mirror lines, and each of them has 6, 3 and 2 mirror lines, respectively. Hence, the signature of this pattern is $*632$.



(a) *632 **(b)** 3*3

Figure 5. Mirror lines and rotations

Gyrations. If the pattern has gyrations (rotations) acted on it, we fix each rotation center, and denote it with the number of its order, which could be $2, 3, 4$ or 6, as we discussed earlier. Also remember that a rotation center cannot be on a mirror line! If the pattern has both mirror lines and gyrations, we put gyration signature in front of mirror line ones. In the case our Fig 5b, we one rotation center, which is denoted 3, and one intersecting point, which is $*3$. So the signature is $3 * 3$.

Miracles. Apart from mirror lines and gyrations, a pattern is also likely to have miracle, which is for glide reflections. The symbol to indicate a miracle is $\times$. Recall that a glide reflection is composed of a reflection and a translation. If there are two glide reflections from two kinds of mirror lines, then we count them as two, and denote $\times\times$. An example of this is Fig 6a, as we can see there are two different mirror lines, which give rise to two different glide reflections, so the signature is $\times\times$.

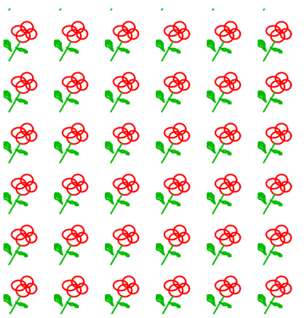
(a) $\times\times$ **(b)** $\circ$

Figure 6. Miracles and wonders

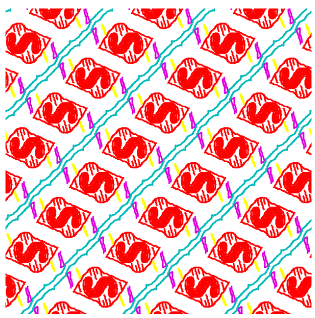
Wonders. Lastly, we have patterns that have translations only, no mirror lines, gyrations, or miracles. Notice that every wallpaper pattern has translations acted on it, but we separate the type of patterns which only contain translations, and we write $\circ$ for patterns of this type. In Fig 6b, we have a pattern consisting of only translations, and therefore denoted as $\circ$.

Since there are 17 wallpaper groups, there are 17 correspondent orbifold signatures: $\circ, 2222, **,$ $22*, 22\times, 2*22, 333, *333, 3*3, \times\times, *\times, *2222, 442, *442, 4*2, 632,$ and $*632$. We will explain why in the section 3. We will next 17 different types of patterns and we will explore it more when we get to the Magic Theorem.

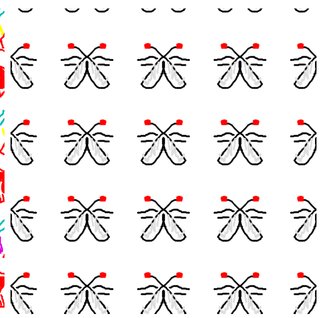
3. **Notations of the wallpaper patterns.** Here, we adopt two notations, one is what we just introduced as orbifold signature invented by John H. Conway (Chapter 1 and 2, [2]), and the alternative one is Hermann–Mauguin notation notation (IUC notation) from International Union of Crystallography. Both of them are listed in the following examples, which are created on *Wallpaper Symmetry*[2].



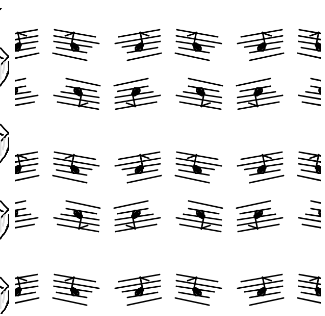
(a) $p1$ (O)



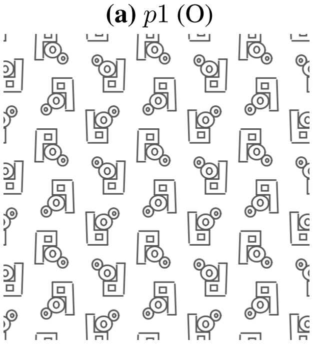
(b) $p2$ (2222)



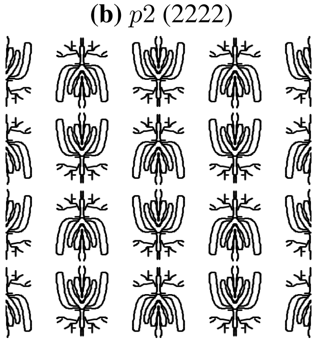
(c) pm $(**)$



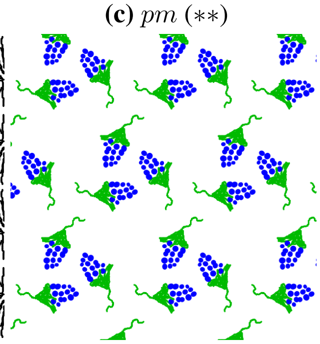
(d) pmg $(22*)$



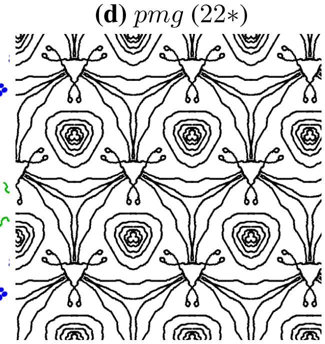
(e) pgg $(22\times)$



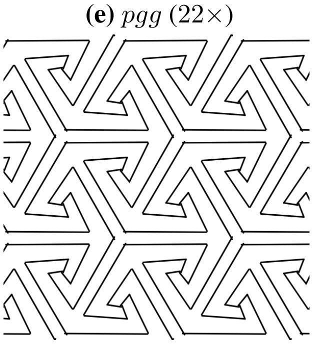
(f) cmm $(2*22)$



(g) $p3$ (333)

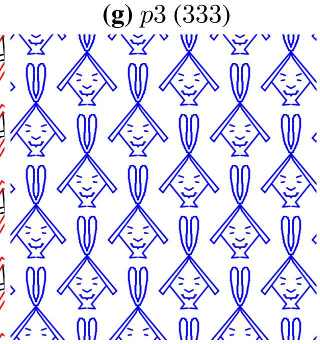


(h) $p3m1$ $(*333)$

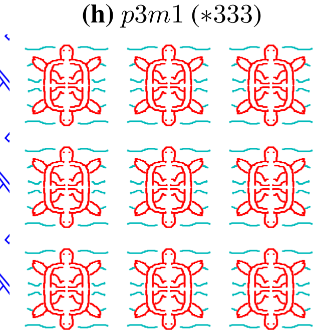


(i) $p31m$ $(3*3)$

(j) pg $(\times\times)$

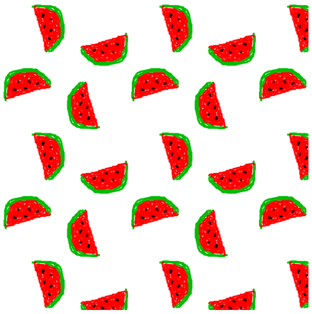


(k) cm $(*\times)$

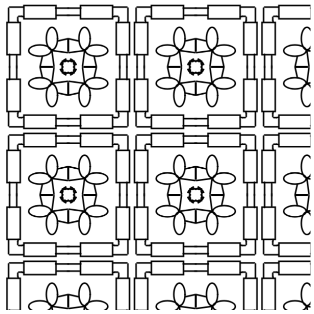


(l) pmm $(*2222)$

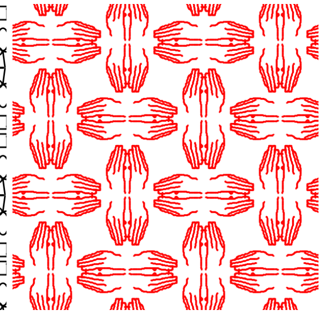
[2]http://math.hws.edu/eck/js/symmetry/wallpaper.html



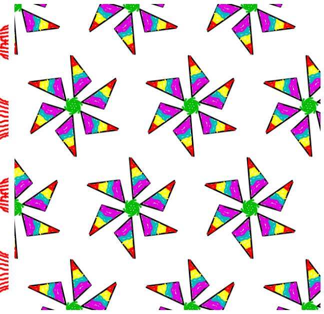
(m) $p4$ (442)



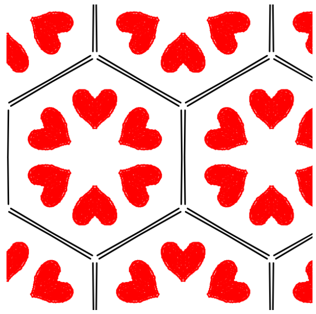
(n) $p4m$ ($*442$)



(o) $p4g$ ($4*2$)



(p) $p6$ (632)



(q) $p6m$ ($*632$)

Figure 7. The 17 wallpaper patterns

4. **Properties of wallpaper groups.** Every wallpaper pattern has a translation subgroup that acts on the lattices. There is one wallpaper group $p1$, which only has translations. There are two subgroups we are interested here (p 452 - 453, [1]).

Theorem 5. *The subgroups of translations of wallpaper groups are isomorphic to $Z \oplus Z$.*

Proof. Let T be the subgroup of all translations. Suppose the units of translation in two directions are $\vec{x}$ and $\vec{y}$, then any translation can be obtained by $t\,\vec{x} + s\,\vec{y}$. Let function $f : T \to Z \oplus Z$ be $t\,\vec{x} + s\,\vec{y} \to (t,\, s)$. Notice that $t_1\,\vec{x} + s_1\,\vec{y} = t_2\,\vec{x} + s_2\,\vec{y}$ if and only if $t_1 = t_2$ and $s_1 = s_2$ because of linear independence of $\vec{x}$ and $\vec{y}$. Then we can easily see that f is one-to-one and onto. Besides, $f((t_1\,\vec{x} + s_1\,\vec{y}) + (t_2\,\vec{x} + s_2\,\vec{y})) = (t_1 + t_2,\, s_1 + s_2)$ implies f is an isometry between T and $Z \oplus Z$, and so they are isomorphic. □

Theorem 6. *The translation subgroup is the center of a symmetry group.*

Proof. In last section, we showed that both rotations and reflections are linear transformations, so we can write a rotation as $R(u) = A_1 u + v_1$ and a reflection as $L(u) = A_2 u + v_2$, where A_1 and A_2 are 2×2 matrices. Similarly, a translation is in form of $T(u) = u + v_3$ and a glide reflection is $G(u) = A_2 u + v_4$. It follows that $R(u)$, $L(u)$ and $G(u)$ do not commute with one another because matrices are non-commutative, whereas $T(u)$ commute with each of them. Hence, the center of the symmetry group is the translation subgroup. □

Some wallpaper groups contain rotations, and some do not. How many of them contain rotations? And, how many types of rotation are there in one group?

Theorem 7. *The subgroup of a rotation of order n around a fixed point is isomorphic to $\mathbb{Z}_n$.*

Proof. Let R denote the rotation subgroup. It is not hard to see that R is cyclic, so $R = \{\frac{2\pi k}{n}; k = 0, 1, \ldots, n-1\}$. We can define a one-to-one correspondence $\phi : R \implies \mathbb{Z}_n$ as $\phi(\frac{2\pi k}{n}) = k$. We then need to verify it is an isometry. Since $\phi\,(\frac{2\pi k_1}{n} + \frac{2\pi k_2}{n}) = k_1 + k_2$, it shows that ϕ is an isometry. Hence, the subgroup of rotation is isomorphic to $\mathbb{Z}_n$. $\square$

At the first look, we realize that the order of rotation in a wallpaper group divides 2π. But there seem to be many options from the divisors of 2π; in fact, the following theorem tells us there are only a few possible orders. We provide a geometric sketch of the proof, and the details can be found in (p 60-61, [5]) for those interested.

Theorem 8. *The only possible orders of rotations in a wallpaper pattern are* 1, 2, 3, 4 *and* 6. *This fact is also known as the crystallographic restriction.*

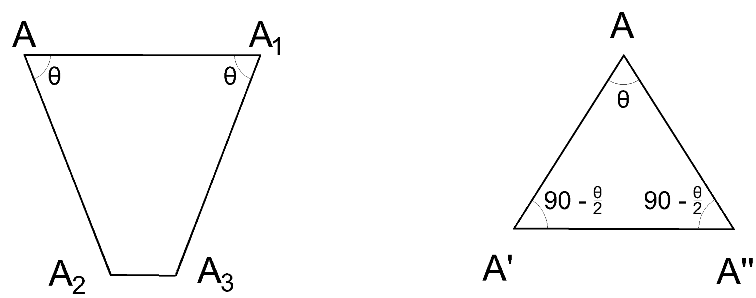


Figure 8. Orders of rotation: $n \leq 5$ (left), $n > 6$ (right)

Proof. Before we start, it is important to know that any symmetry operation such as a translation or rotation will map a rotation center of order n to another rotation center of order n, which is the same lattice point. We use Fig 8 to prove by elimination.

We first show that $n > 6$ does not hold. Please see the right figure of Fig 8. We have a rotation center A, and assume $\theta = \frac{360°}{n}$, where n is the order of rotation. And assume A' is a closest rotation center point of order n near A, then for any other rotation center point B of the same order, $|AB| \geq |AA'|$. Now, by a rotation of θ, A' is mapped to A''. So $|AA''| \geq |AA'|$, which is equivalent to $\angle\theta \geq \angle A''$. We know that $\angle A'' = \angle A' = 90° - \frac{\theta}{2}$; thus, $\theta \geq 90° - \frac{\theta}{2}$, which implies $\theta \geq 60°$. So it follows that $n \leq 6$, and now the case $n > 6$ is eliminated.

But for $n = 1, 2, 3, 4, 5$ or 6, we will show that $n = 5$ is impossible using the similar argument. Please see the left figure of Fig 8. A rotation center of order n A is chosen, and A_1 is a closest rotation center around A. Let us map A_1 to A_2 and A to A_3 by a rotation of θ around A and A_1, respectively. Similarly, if A_2 and A_3 are distinct, one can argue that $|A_2A_3| \geq |AA_1|$ by the minimality of $|AA_1|$. This is true if either $\angle A = \theta \geq \angle A_2 = 180° - \theta$ or A_2 and A_3 are the same point. As a result, $\theta \leq 90°$ or $\theta = 60°$, so $\theta \neq 72°$. What we just did rules out the case of $n = 5$. So we are left with the cases $n = 1, 2, 3, 4$ or 6 to be verified by other criteria.

$n = 1$ is the case of a trivial group, since rotation of order 1 is an identity. But further argument to guarantee the possibilities of all $n = 2, 3, 4$ or 6 requires study of parallelogram generating region of plane patterns, which we will not discuss here. $\square$

3. THE MAGIC THEOREM

Now, it is our time to reveal the Magic Theorem by John H. Conway in chapters 3 and 6 from [2]. It is extremely useful in terms of understanding existence of 17 types of wallpaper groups. Before we introduce the theorem itself, we need to know the "cost" of each operation.

1. **Cost of operations.** We assign a value, called cost, to each type of mirror line, gyration, miracle and wonder, as listed in the following table (P 29, [2]). In order to calculate the cost of one wallpaper patter, we need to find its orbifold signature.

Symbol	Cost ($)	Symbol	Cost ($)
$\circ$	2	$*$ or $\times$	1
2	$1/2$	2	$1/4$
3	$2/3$	3	$1/3$
4	$3/4$	4	$3/8$
5	$4/5$	5	$2/5$
6	$5/6$	6	$5/12$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	$(N-1)/N$	N	$(N-1)/2N$
∞	1	∞	$1/2$

Table 2. The cost table

Let us practice to calculate the example in Fig 5a: the total cost $= \frac{5}{6} + \frac{2}{3} + \frac{1}{2} = 2$, and in Fig 5b: the total cost $= \frac{2}{3} + 1 + \frac{1}{3} = 2$. In both cases, we obtain a cost of \$2, and if you try it on Fig 6a and 6b. you will still get the same result. This is not a coincidence, but a result from the Magic Theorem.

The following theorem can be found on p 30 in [2].

Theorem 9 (The Magic Theorem for plane repeating patterns). *The signatures of plane repeating patterns are precisely those with total cost* \$2.

Next, we will list all the possibilities of signatures according to the Magic Theorem to all the plane wallpaper pattern types. Now, we can prove the theorem:

Theorem. *There are exactly* 17 *wallpaper groups in* $\mathbb{R}^2$.

Proof. Using the table 2, we list all the possibilities by classifying into three categories: patterns with no mirror lines, with one mirror line and with more than one mirror lines.

Case one (no mirror lines): we can have translation, rotations, glide reflections or a combination of them that add up to a cost of \$2: $\circ$, $\times\times$, $22\times$, 2222, 333, 442, and 632. That is 7 types.

Case two (one mirror line): we can have one mirror line combined with rotations or glide reflections. There are $*\times$, $22*$, $4*2$, and $3*3$, 4 types.

Case three (more than one mirror line): we can have a combination of at least two mirror lines together with rotations: $**$, $2*22$, $*2222$, $*333$, $*442$, and $*632$, which is 6 types.

If we add up all the types in three cases together, we get exactly 17 wallpaper patterns, as desired. $\square$

For those interested in why Magic Theorem works, please refer to chapter 6 in book [2], and they will use Euler's characteristic to explain it.

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