The Lattice Approach for Finding Conserved Quantities of Dynamical Systems

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Abstract

Conserved quantities can tell us many properties of a dynamical system. However, such desired quantities often do not exist or cannot be found systematically. In this research project, we investigated a systematic approach to find conserved quantities for polynomial dynamical systems in $\mathbb{R}^2$ or $\mathbb{R}^3$, called the "lattice approach". For quadratic 2D systems and some chaotic 3D systems, this approach helps us derive conditions for existence of polynomial or analytic conserved quantities. Our approach converts the original problem to solving a sequence of linear recurrences. As we will see, under certain assumptions, solutions to those linear recurrences will give us polynomial or analytic conserved quantities.

Acknowledgement

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Chapter 1

Introduction

Consider a general dynamical system of $\boldsymbol{x}(t)$ given by

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{g}(\boldsymbol{x}), \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \end{cases} \tag{1.1}$$

where the dot indicates differentiation with respect to t. In the following, we will assume basic knowledge of dynamical systems. See for example [4] for more details on this subject. We are only interested in dynamical systems that have a unique solution on some neighborhood, thus we recall the following well-known existence and uniqueness theorem.

Definition 1. *If a function $\boldsymbol{g}$ is continuous on a subset $U \in \mathbb{R}^n$, we say that $\boldsymbol{g}$ belongs to the class of continuous functions on U, written as $\boldsymbol{g} \in C(U)$.*

Definition 2. *We say that $\boldsymbol{g} \in C^1(U)$ if $\boldsymbol{g} : U \to \mathbb{R}^n$ is continuously differentiable on U.*

Theorem 1 (Local existence-uniqueness theorem)**.** *Let $U \in \mathbb{R}^n$ be open and $\boldsymbol{x}_0 \in U$. For system* (1.1)*, if $\boldsymbol{g} \in C^1(U)$, then there exists some $\delta > 0$ such that a unique solution $\boldsymbol{x}(t)$ exists on $I = [t_0 - \delta,\, t_0 + \delta]$.*

Definition 3. *A conserved quantity $\psi(\boldsymbol{x}, t) : U \times I \to \mathbb{R}$ of the system* (1.1) *is a differentiable function which satisfies $\dfrac{d\psi}{dt} = 0$ or equivalently, $\psi(\boldsymbol{x}, t) = C$ on any solution of* (1.1) *for all $t \in I$ with the constant $C := \psi(\boldsymbol{x}(t_0), t_0)$. A conserved quantity $\psi(\boldsymbol{x}(t), t)$ is said to be time-independent if it does not depend on t explicitly, i.e., $\psi(\boldsymbol{x}, t) = \psi(\boldsymbol{x}(t))$; otherwise it is called time-dependent.*

In this work, we are only interested in nontrivial conserved quantities, i.e., non-constant function $\psi(\boldsymbol{x}, t)$. We also recall definitions on a special class of dynamical systems and certain limiting sets of a dynamical system.

Definition 4 (Hamiltonian System)**.** *A Hamiltonian system in $\mathbb{R}^{2n}$ is of the form*

$$\dot{\boldsymbol{x}} = \frac{\partial H}{\partial \boldsymbol{y}}$$
$$\dot{\boldsymbol{y}} = -\frac{\partial H}{\partial \boldsymbol{x}},$$

where $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$ and $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a differentiable function called the "Hamiltonian".

Definition 5 (ω and α limit set)**.** *Let Γ be a trajectory of system* (1.1)*. An $\omega-$limit set of Γ is $\omega(\Gamma) = \{\boldsymbol{p} \in U :$ there exists a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} = \boldsymbol{p}\}$. Similarly, an $\alpha-$limit set is $\omega(\Gamma) = \{\boldsymbol{p} \in U :$ there exists a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} = \boldsymbol{p}\}$.*

In the next chapters, we will study conserved quantities of general quadratic systems and some chaotic 3D systems.

Chapter 2

Quadratic 2D Systems

A general dynamical system in $\mathbb{R}^2$ can be written as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{g}(\boldsymbol{x}), \\ \boldsymbol{x}(t_0) &= \boldsymbol{x}_0, \end{aligned} \tag{2.1}$$

where $\boldsymbol{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is the unknown functions and $\boldsymbol{g}(\boldsymbol{x}) = \begin{bmatrix} g_1(x,\, y) \\ g_2(x,\, y) \end{bmatrix}$ is some vector-valued function.

By Theorem 1, system (2.1) has a unique solution on some open subset $I \subset \mathbb{R}$. In particular, a general quadratic 2D system, (2.1) has the form

$$\begin{cases} \dot{x} &= Ax + By + Cx^2 + Dxy + Ey^2, \\ \dot{y} &= Fx + Gy + Hx^2 + Ixy + Jy^2, \end{cases} \tag{Q2D}$$

where A, B, C, D, E, F, G, H, I and J are constants.

Examples of such systems include Hamiltonian systems[1]

$$\begin{aligned} \dot{x} &= Ax + By + Cx^2 - 2Jxy + Ey^2, \\ \dot{y} &= Fx - Ay + Hx^2 - 2Cxy + Jy^2. \end{aligned} \tag{2.2}$$

and the Lotka–Volterra system which is known as an example of “Poission system”,

$$\begin{aligned} \dot{x} &= ax - bxy, \\ \dot{y} &= sxy - cy. \end{aligned} \tag{2.3}$$

The purpose of this section is to derive time-dependent conserved quantity $\psi(x, y, t)$ for such systems (Q2D) in the form

$$\psi(x, y, t) = f(x, y)\, e^{-kt}, \tag{2.4}$$

where $f(x, y)$ is analytic around the origin[2] and $k \in \mathbb{R}$. By the definition of conserved quantity, we know that $\psi(x, y, t) = C$ on any solution of (Q2D) for some constant C depending on the initial condition. Notice that for $k < 0$, $f(x, y) \to 0$ as $t \to \infty$. This reveals that the solution to (Q2D) tends to the zero set of $f(x, y)$, i.e., $\omega-$limit set, for sufficiently large t.

[1] In the Theorem 6 of appendix we show that all quadratic Hamiltonian systems in $\mathbb{R}^2$ must have the form (2.2).

[2] W.L.O.G., under appropriate change of variables, $f(x, y)$ can be made analytic around the origin.

Assuming a conserved quantity of this form exists, then we will derive the necessary conditions and show that $f(x, y)$ exists as a polynomial under certain conditions. Since $f(x, y)$ is analytic around the origin by hypothesis, we can write

$$f(x, y) = \sum_{m=0,n=0}^{\infty} C_{m,n}\, x^m y^n. \tag{2.5}$$

Then it follows that on any solution of (Q2D), $0 = \frac{\mathrm{d}\psi}{\mathrm{d}t} = \left(\frac{\mathrm{d}f}{\mathrm{d}t} - kf\right) e^{-kt}$. This implies $\frac{\mathrm{d}f}{\mathrm{d}t} - kf = 0$, and so by chain rule, we obtain $0 = f_x g_1(x,\, y) + f_y g_2(x,\, y) - kf$. Differentiating (2.5) and substituting (Q2D) gives:

$$\begin{aligned} 0 = &\sum_{m=0,n=0}^{\infty} [(m+1)C_{m+1,n}\, x^m y^n (Ax + By + Cx^2 + Dxy + Ey^2) \\ &+ (n+1)C_{m,n+1}\, x^m y^n (Fx + Gy + Hx^2 + Ixy + Jy^2)] - k \sum_{m=0,n=0}^{\infty} C_{m,n}\, x^m y^n \end{aligned} \tag{2.6}$$

In the next step, we simplify (3.4) by separating terms that have a factor of $x^l y^m$ for some $l, m \geq 2$ and grouping those that do not.

$$\begin{aligned} 0 = &\sum_{n=2}^{\infty}[AC_{1,n} + F(n+1)C_{0,n+1}]xy^n + \sum_{m=0}^{\infty}[A(m+1)C_{m+1,0} + FC_{m,1}]x^{m+1} \\ &+ \sum_{m=0}^{\infty}[A(m+1)C_{m+1,1} + 2FC_{m,2}]x^{m+1}y + \sum_{m=2,n=2}^{\infty}[AmC_{m,n} + F(n+1)C_{m-1,n+1}]x^m y^n \\ &+ \sum_{m=2}^{\infty}[B(m+1)C_{m+1,0} + GC_{m,1}]x^m y + \sum_{n=0}^{\infty}[BC_{1,n} + G(n+1)C_{0,n+1}]y^{n+1} \\ &+ \sum_{n=0}^{\infty}[2BC_{2,n} + G(n+1)C_{1,n+1}]xy^{n+1} + \sum_{m=2,n=2}^{\infty}[B(m+1)C_{m+1,n-1} + GnC_{m,n}]x^m y^n \\ &+ \sum_{m=0}^{\infty}[C(m+1)C_{m+1,0} + HC_{m,1}]x^{m+2} + \sum_{m=0}^{\infty}[C(m+1)C_{m+1,1} + 2HC_{m,2}]x^{m+2}y \\ &+ \sum_{m=2,n=2}^{\infty}[C(m-1)C_{m-1,n} + H(n+1)C_{m-2,n+1}]x^m y^n \\ &+ \sum_{n=0}^{\infty}[DC_{1,n} + I(n+1)C_{0,n+1}]xy^{n+1} + \sum_{m=1}^{\infty}[D(m+1)C_{m+1,0} + IC_{m,1}]x^{m+1}y \\ &+ \sum_{m=2,n=2}^{\infty}[DmC_{m,n-1} + InC_{m-1,n}]x^m y^n \\ &+ \sum_{n=0}^{\infty}[EC_{1,n} + J(n+1)C_{0,n+1}]y^{n+2} + \sum_{n=0}^{\infty}[2EC_{2,n} + J(n+1)C_{m,1}]xy^{n+2} \\ &+ \sum_{m=2,n=2}^{\infty}[E(m+1)C_{m+1,n-2} + J(n-1)C_{m,n-1}]x^m y^n - k\sum_{m=0,n=0}^{\infty} C_{m,n}\, x^m y^n \\ = &\underbrace{[(A-k)C_{1,0} + FC_{0,1}]}_{\text{Q2D-1a}}\, x + \underbrace{[BC_{1,0} + (G-k)C_{0,1}]}_{\text{Q2D-1b}}\, y + \underbrace{[2BC_{2,0} + (G+A-k)C_{1,1} + DC_{1,0}}_{\text{Q2D-2}} \end{aligned}$$

$$\underbrace{+IC_{0,1} + 2FC_{0,2}}\,xy + \sum_{m=2}^{\infty} \underbrace{[(Am-k)C_{m,0} + FC_{m-1,1} + C(m-1)C_{m-1,0} + HC_{m-2,1}]}_{\text{Q2D-3}}\,x^m$$

$$+\sum_{n=2}^{\infty} \underbrace{[BC_{1,n-1} + (Gn-k)C_{0,n} + EC_{1,n-2} + J(n-1)C_{0,n-1}]}_{\text{Q2D-4}}\,y^n + \sum_{n=2}^{\infty} \underbrace{[(A+Gn-k)C_{1,n}}$$

$$\underbrace{+F(n+1)C_{0,n+1} + 2BC_{2,n-1} + (D+J(n-1))C_{1,n-1} + InC_{0,n} + 2EC_{2,n-2}]}_{\text{Q2D-5}}\,xy^n$$

$$+\sum_{m=2}^{\infty} \underbrace{[(Am+G-k)C_{1,n} + 2FC_{m-1,2} + B(m+1)C_{m+1,0}\,(C(m-1)+I)C_{m-1,1}}_{\text{Q2D-6}}$$

$$\underbrace{+2HC_{m-2,2} + DmC_{m,0}]}\,x^m y + \sum_{m=2,n=2}^{\infty} \underbrace{[(Am+Gn-k)C_{m,n} + F(n+1)C_{m-1,n+1}}$$

$$\underbrace{+B(m+1)C_{m+1,n-1} + (C(m-1)+In)C_{m-1,n} + H(n+1)C_{m-2,n+1}}_{\text{Q2D-7}}$$

$$\underbrace{+(Dm+J(n-1))C_{m,n-1} + E(m+1)C_{m+1,n-2}]}\,x^m y^n. \tag{2.7}$$

Next, we will systematically relate (1) – (7) by the “lattice approach”.

2.1 Equations and Lattices

Since the analytic condition (2.5) holds in the neighborhood around the origin, all seven expressions of (2.7) must vanish. To study the dependency of these expressions, it helps to depict them in the following diagrams. Since the indices m, n are natural numbers, it is natural to represent them as points on a “lattice”. In each lattice diagram, the dots represent the coefficients $C_{i,j}$, and the lines represent their relations given by the expresions. As m increases, the lattice will move one unit to the right and as n increases, it will move ont unit upwards. The solid lines in the diagrams show the relations involving the lowest order indices and the color dotted lines show the typical relation as n or m increases.

(Q2D-1a) $(A-k)C_{1,0} + FC_{0,1} = 0$

(Q2D-1b) $BC_{1,0} + (G-k)C_{0,1} = 0$

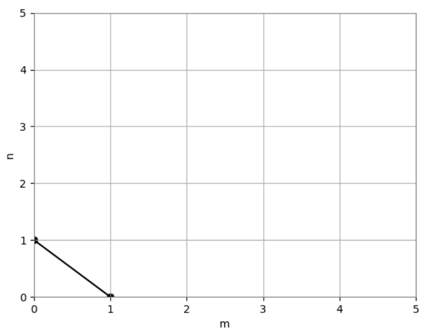


Figure 2.1: Lattice diagram Q2D-1a and Q2D-1b

(Q2D-2) $2BC_{2,0} + (G + A - k)C_{1,1} + DC_{1,0} + IC_{0,1} + 2FC_{0,2} = 0$

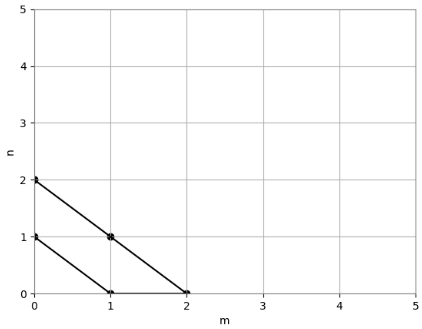


Figure 2.2: Lattice diagram Q2D-2

(Q2D-3) $(Am - k)C_{m,0} + FC_{m-1,1} + C(m - 1)C_{m-1,0} + HC_{m-2,1} = 0,\ m \geq 2$

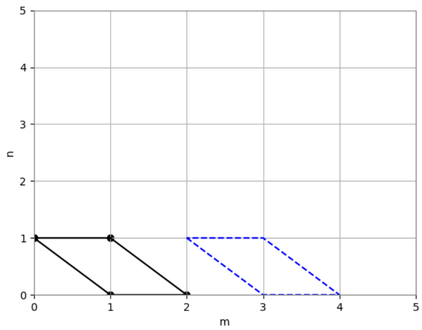


Figure 2.3: Lattice diagram Q2D-3

(Q2D-4) $BC_{1,n-1} + (Gn - k)C_{0,n} + EC_{1,n-2} + J(n - 1)C_{0,n-1} = 0,\ n \geq 2$

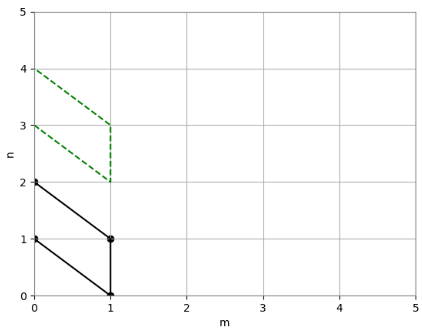


Figure 2.4: Lattice diagram Q2D-4

(Q2D-5)

$(A+Gn-k)C_{1,n} + F(n+1)C_{0,n+1} + 2BC_{2,n-1} + [D+I(n-1)]C_{1,n-1} + InC_{0,n} + 2EC_{2,n-2} = 0,\ n \geq 2$

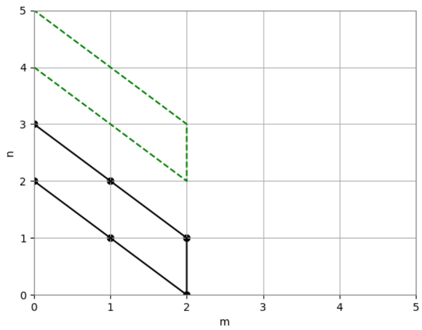


Figure 2.5: Lattice diagram Q2D-5

(Q2D-6) $(Am+G-k)C_{m,1} + 2FC_{m-1,2} + B(m+1)C_{m+1,0} + [C(m-1)+I]C_{m-1,1}\, 2HC_{m-2,2} + DmC_{m,0} = 0,\; m \geq 2$

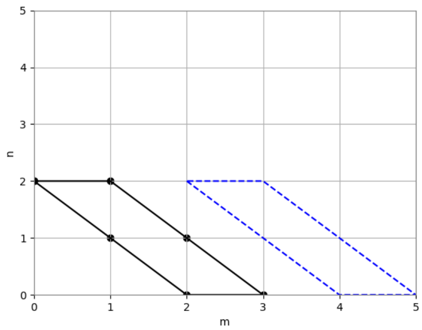


Figure 2.6: Lattice diagram Q2D-6

(Q2D-7)

$$(Am + Gn - k)C_{m,n} + F(n+1)C_{m-1,n+1} + B(m+1)C_{m+1,n-1} + [C(m-1) + In]C_{m-1,n}$$
$$+H(n+1)C_{m-2,n+1} + [Dm + J(n-1)]C_{m,n-1} + E(m+1)C_{m+1,n-2} = 0,\; m, n \geq 2.$$

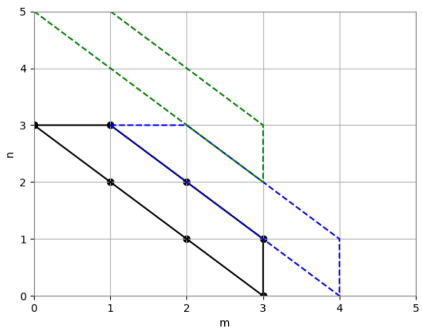


Figure 2.7: Lattice diagram Q2D-7

2.2 Solutions in equivalent matrix form

For convenience of analysis, we introduce some notations and rewrite equations (Q2D-1a) through (Q2D-7) in matrix form. Each matrix equation involves the coefficients with the same power of the subsequent binomial terms. If a solution to the matrix equations exists, then the system (Q2D) has a conserved quantity. Depending on the values of parameters in system Q2D, there are three main cases: a solution to the matrix equations might not exist, or the solution exists with a finite number of nonzero coefficients, or analytic if there are an infinite number of nonzero coefficients. The latter two cases imply existence of a polynomial or analytic conserved quantity for (Q2D). Let M denote the sum of powers of x and y.
$M = 1$: (Q2D-1a) and (Q2D-2) imply

$$\begin{bmatrix} A-k & F \\ B & G-k \end{bmatrix} \begin{bmatrix} C_{1,0} \\ C_{0,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$M = 2$: (Q2D-2), (Q2D-3) with $m = 2$ and (Q2D-4) with $n = 2$ imply

$$\begin{bmatrix} 2A-k & F & 0 \\ 2B & G+A-k & 2F \\ 0 & B & 2G-k \end{bmatrix} \begin{bmatrix} C_{2,0} \\ C_{1,1} \\ C_{0,2} \end{bmatrix} = \begin{bmatrix} -C & -H \\ -D & -I \\ -E & -J \end{bmatrix} \begin{bmatrix} C_{1,0} \\ C_{0,1} \end{bmatrix}.$$

$M = 3$: (Q2D-3) with $m = 3$, (Q2D-4) with $n = 3$, (Q2D-5) with $n = 2$ and (Q2D-6) with $m = 2$ imply

$$\begin{bmatrix} 3A-k & F & 0 & 0 \\ 3B & G+2A-k & 2F & 0 \\ 0 & 2B & 2G+A-k & 3F \\ 0 & 0 & B & 3G-k \end{bmatrix} \begin{bmatrix} C_{3,0} \\ C_{2,1} \\ C_{1,2} \\ C_{0,3} \end{bmatrix} = \begin{bmatrix} -2C & -H & 0 \\ -2D & -(C+I) & -2H \\ -2E & -(D+J) & -2I \\ -E & 0 & -2J \end{bmatrix} \begin{bmatrix} C_{2,0} \\ C_{1,1} \\ C_{0,2} \end{bmatrix}.$$

$\vdots$

M: (Q2D-5) with $n = M-1$, (Q2D-6) with $m = M-1$ and (Q2D-7) with $m = 2, \dots, M-2$, $n = M-2, \dots, 2$.

$$(P_M - kI)\boldsymbol{v}_M = Q_M \boldsymbol{v}_{M-1} \tag{2.8}$$

where $\boldsymbol{v}_M = \begin{bmatrix} C_{M,0} \\ C_{M-1,1} \\ C_{M-2,2} \\ \dots \\ C_{2,M-2} \\ C_{1,M-1} \\ C_{0,M} \end{bmatrix}$ and $\boldsymbol{v}_{M-1} = \begin{bmatrix} C_{M-1,0} \\ C_{M-2,1} \\ \vdots \\ C_{2,M-3} \\ C_{1,M-2} \\ C_{0,M-1} \end{bmatrix}$,

$P_M - kI =$

$$\begin{bmatrix} MA-k & F & 0 & \dots & 0 & & \\ MB & (M-1)A+G-k & 2F & 0 & \dots & 0 & \\ 0 & (M-1)B & (M-2)A+2G-k & 3F & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & & & & \\ 0 & \dots & 0 & 3B & 2A+(M-2)G-k & (M-1)F & 0 \\ 0 & \dots & 0 & 0 & 2B & A+(M-1)G-k & MF \\ 0 & \dots & 0 & 0 & 0 & B & MG-k \end{bmatrix}$$

and $Q_M =$

$$\begin{bmatrix} -C & -H & 0 & \dots & 0 & & & \\ -D & -[(M-2)C+I] & -2H & 0 & \dots & 0 & & \\ -(M-1)E & -[(M-2)D+J] & -[(M-3)C+2I] & -3H & & & & \\ \ddots & \ddots & & & & & & \\ 0 & \dots & 0 & -3E & -[2D+(M-3)J] & -[C+(M-2)I] & -(M-1)H \\ 0 & \dots & 0 & 0 & -2E & -[D+(M-2)J] & -(M-1)I \\ 0 & \dots & 0 & 0 & 0 & -E & -(M-1)J \end{bmatrix}$$

Using the same notation, it is easy to see the following relations:

$$\begin{aligned} (P_1 - kI)\boldsymbol{v}_1 &= 0 \\ (P_2 - kI)\boldsymbol{v}_2 &= Q_2\boldsymbol{v}_1 \\ (P_3 - kI)\boldsymbol{v}_3 &= Q_3\boldsymbol{v}_2 \\ &\vdots \\ (P_n - kI)\boldsymbol{v}_n &= Q_n\boldsymbol{v}_{n-1} \\ (P_{n+1} - kI)\boldsymbol{v}_{n+1} &= Q_{n+1}\boldsymbol{v}_n \\ &\vdots \end{aligned} \tag{2.9}$$

2.3 Polynomial conserved quantities

In this section, we seek to find polynomial conserved quantities to system Q2D. Before we deduce any conditions on existence of such conserved quantities, let us motivate with an example given by the Hamiltonian system

$$\begin{aligned} \dot{x} &= Ax + By + Cx^2 - 2Jxy + Ey^2, \\ \dot{y} &= Fx - Ay + Hx^2 - 2Cxy + Jy^2. \end{aligned} \tag{2.10}$$

From Theorem 6 in the appendix, we have the conserved quantity

$$\psi(x,\, y) = -\frac{F}{2}x^2 + Axy + \frac{B}{2}y^2 - \frac{H}{3}x^3 + Cx^2y - Jxy^2 + \frac{E}{3}y^3. \tag{2.11}$$

Indeed, we can verify that the coefficients of (2.11) satisfy the lattice equations (2.9) as follows

$$(P_1 - kI)\boldsymbol{v}_1 = 0 \iff$$

$$\begin{bmatrix} A & F \\ B & -A \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(P_2 - kI)\boldsymbol{v}_2 = Q_2\boldsymbol{v}_1 \iff$$

$$\begin{bmatrix} 2A & F & 0 \\ 2B & 0 & 2F \\ 0 & B & -2A \end{bmatrix} \begin{bmatrix} -\frac{F}{2} \\ A \\ \frac{B}{2} \end{bmatrix} = \begin{bmatrix} -C & -H \\ 2J & 2C \\ -E & -J \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(P_3 - kI)\boldsymbol{v}_3 = Q_3\boldsymbol{v}_2$$

$$\begin{bmatrix} 3A & F & 0 & 0 \\ 3B & A & 2F & 0 \\ 0 & 2B & -A & 3F \\ 0 & 0 & B & -3A \end{bmatrix} \begin{bmatrix} -\frac{H}{3} \\ C \\ -J \\ \frac{E}{3} \end{bmatrix} = \begin{bmatrix} -2C & H & 0 \\ 4J & C & -2H \\ -2E & J & 4C \\ 0 & -E & -2J \end{bmatrix} \begin{bmatrix} -\frac{F}{2} \\ A \\ \frac{B}{2} \end{bmatrix}$$

$$(P_4 - kI)\boldsymbol{v}_4 = Q_4\boldsymbol{v}_3$$

$$0 = \begin{bmatrix} -3C & -H & 0 & 0 \\ 6J & 0 & -2H & 0 \\ -3E & 3J & 3C & -3H \\ 0 & -2E & 0 & 6C \\ 0 & 0 & -E & -3J \end{bmatrix} \begin{bmatrix} -\frac{H}{3} \\ C \\ -J \\ \frac{E}{3} \end{bmatrix}$$

Note that $\ker Q_n \neq \emptyset$ is a necessary condition for nonzero $\boldsymbol{v}_n$. Thus, this illustrates the necessary condition (2.9) for conserved quantities. In general, it is difficult to solve these lattice equations. Instead, we will derive a subset of solutions to the lattice equations below.

First, we illustrate a specific case with homogeneous polynomial conserved quantities. Here it is required that $\boldsymbol{v}_n \neq 0$ and $\boldsymbol{v}_k = 0$ for all $k \neq n$.
From equation (2.8) with $M = n + 1$, we find

$$0 = (P_{n+1} - kI)\boldsymbol{v}_{n+1} = Q_{n+1}\boldsymbol{v}_n,$$

we can deduce that $\boldsymbol{v}_n \in \ker Q_{n+1}$. Moreover, from equation (2.8) with $M = n$,

$$(P_n - kI)\boldsymbol{v}_n = Q_n\boldsymbol{v}_{n-1} = 0,$$

it follows that $\boldsymbol{v}_n \in \ker(P_n - kI)$. Thus, we have the following theorem.

Theorem 2. *If* $\ker Q_{n+1} \cap \ker(P_n - kI) \neq \emptyset$ *for some* $k \in \mathbb{R}$*, then system* (Q2D) *has conserved quantity of the form* (2.4) *with an* n^{th} *order homogeneous polynomial* $f(x, y)$ *where the coefficients are defined by* $\boldsymbol{v}_n \in \ker Q_{n+1} \cap \ker(P_n - kI)$*.*

In fact, Theorem 2 can be generalized to polynomial conserved quantities. Specifically, a sufficient condition is to have $f(x, y)$ in (2.4) as an $n-$th order polynomial, and the details are stated in the next theorem.

Theorem 3. *If* $\boldsymbol{v}_n \in \ker Q_{n+1} \cap \ker(P_n - kI) \neq \emptyset$ *for some* $k \in \mathbb{R}$ *and* $\boldsymbol{v}_i \in \ker Q_{i+1} \cap \ker P_i - kI$ *for all* $i = 1, \ldots, n-1$*, then system* (Q2D) *has a conserved quantity of the form* (2.4) *with an* n^{th} *order polynomial* $f(x, y)$*, where the coefficients are defined by* $\boldsymbol{v}_i$ *for* $i = 1, \ldots, n-1$*.*

This theorem removes the restriction of $f(x, y)$ being only homogeneous polynomials. Following from (2.9), this theorem suggests that

$$\begin{aligned} (P_1 - kI)\boldsymbol{v}_1 &= 0 \\ (P_2 - kI)\boldsymbol{v}_2 &= 0 = Q_2\boldsymbol{v}_1 \\ &\vdots \\ (P_n - kI)\boldsymbol{v}_n &= 0 = Q_n\boldsymbol{v}_{n-1} \\ (P_{n+1} - kI)\boldsymbol{v}_{n+1} &= 0 = Q_{n+1}\boldsymbol{v}_n. \end{aligned}$$

For practical purposes, we will introduce an algorithm to find the coefficients of $f(x, y)$ in (2.5), if an nontrivial polynomial conserved quantity exists. Specifically, let us introduce an algorithm to find the intersection of two subspaces A and B and a general algorithm to find coefficients $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of $f(x, y)$ in (2.5), if there is any.

Algorithm 1 FindIntersection(A, B)

$N_A \leftarrow$ rank A, $N_B \leftarrow$ rank B.
Let $\{\boldsymbol{a}_i\}_{i=1}^{N_A}$ be a basis of A and $\{\boldsymbol{b}_i\}_{i=1}^{N_B}$ be a basis of B.
Suppose $\boldsymbol{v} = \sum_{i=1}^{N_A} C_i \boldsymbol{a}_i = \sum_{i=1}^{N_B} D_i \boldsymbol{b}_i$, then compute the kernel of

$$\begin{bmatrix} \boldsymbol{a}_1 & \dots & \boldsymbol{a}_{N_A} & \boldsymbol{b}_1 & \dots & \boldsymbol{b}_{N_B} \end{bmatrix}.$$

Algorithm 2 Find the coefficients for $n-$th order polynomial

$L \leftarrow$ list of vectors
for each $k_i \in$ eigs P_n counting multiplicity **do**
  find the corresponding eigenvector $\boldsymbol{v}^i$
  if $Q_{n+1}\boldsymbol{v}^i = \boldsymbol{0}$ **then**
    L.addToList($\boldsymbol{v}^i$)
  end if
end for
if L is empty **then**
  $\boldsymbol{v}_n = 0$ and there is no polynomial of order n
else
  for each $\boldsymbol{v}^i \in L$ **do**
    $\boldsymbol{v}_n^i \leftarrow \boldsymbol{v}^i$
    for $j = n-1, \dots, 1$ **do**
      $\boldsymbol{v}_j^i \leftarrow$ FindIntersection(ker $(P_j - kI)$, ker Q_{j+1})
    end for
    the solution is defined its $\boldsymbol{v}_1^i, \dots, \boldsymbol{v}_n^i$
  end for
end if

Now, let us illustrate the algorithm by the following example

$$\begin{aligned} \dot{x} &= Cx^2 + Dxy + (-4C - 2D)y^2, \\ \dot{y} &= -2Cx^2 + (-5C - 2D)xy + (-2C - D)y^2, \end{aligned} \tag{2.12}$$

where C, D are nonzero constants.

We want to find a polynomial conserved quantity, say, of order 2. Thus we will have $\boldsymbol{v}_n = 0$ for all $n \geq 3$. We first need to find an eigenvalue for P_2. Since $P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we see that $k = 0$ is the only eigenvalue and its corresponding eigenvector is $\boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for any $x, y, z \in \mathbb{R}$. With $Q_2 = \begin{bmatrix} -2C & 2C & 0 \\ -2D & 4C+2D & 4C \\ 8C+4D & 2C & 10C+4D \\ 0 & 4C+2D & 4C+2D \end{bmatrix}$, let $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and we check that $Q_2\boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

So we set $\boldsymbol{v}_2 = \boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. With $k = 0$, we need to solve

$$P_2 \boldsymbol{v}_2 = Q_2 \boldsymbol{v}_1 \iff 0 = \begin{bmatrix} -C & 2C \\ -D & 5C + 2D \\ 4C + 2D & 2C + D \end{bmatrix} \boldsymbol{v}_1.$$

Since C, D cannot both be zero, then $\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \ker (P_1 - 0I) \cap \ker Q_2$. Therefore, the algorithm implies $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\boldsymbol{v}_n = 0$ for $n = 1$ and $n \neq 3$ and a conserved quantity for system 2.12 is given by

$$\psi(x, y) = x^2 + xy - y^2.$$

Chapter 3

Cubic 2D system

In this chapter, we will study general cubic 2D systems for more interesting results in an analogous approach. Consider the cubic 2D system given by

$$\begin{aligned}\dot{x} &= Ax + By + Cx^2 + Dxy + Ey^2 + Fx^3 + Gx^2y + Hxy^2 + Iy^3, \\ \dot{y} &= Jx + Ky + Lx^2 + Mxy + Ny^2 + Ox^3 + Px^2y + Qxy^2 + Ry^3.\end{aligned} \tag{C2D}$$

Again, we are interested in the so-called "Darboux first interal":

$$\psi(x, y, t) = f(x.y)e^{-\lambda t} = e^{-\lambda t} \sum_{m=0,n=0}^{\infty} C_{m,n}\, x^m y^n. \tag{3.1}$$

We find the derivatves:

$$\frac{\partial f}{\partial x} = \sum_{m=0,n=0}^{\infty} (m+1)C_{m+1,n}\, x^m y^n, \tag{3.2}$$

$$\frac{\partial f}{\partial y} = \sum_{m=0,n=0}^{\infty} (n+1)C_{m,n+1}\, x^m y^n. \tag{3.3}$$

Next, we have the following relation:

$$0 = \frac{\mathrm{d}\psi}{\mathrm{d}t} = e^{-kt}\left(\frac{\mathrm{d}f}{\mathrm{d}t} - kf\right) = e^{-kt}\left(\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} - kf\right). \tag{3.4}$$

Substuting equations (C2D), (3.2) and (3.3) into (3.4) simplies to

$$\begin{aligned}0 =& [\underbrace{BC_{1,0} + (K-\lambda)C_{0,1}}_{\text{C2D-1}}]y + [\underbrace{BC_{1,1} + (2K-\lambda)C_{0,2} + EC_{1,0} + NC_{0,1}}_{\text{C2D-2}}]y^2 \\ &+ \sum_{n=3}^{\infty}[\underbrace{BC_{1,n-1} + (nK-\lambda)C_{0,n} + EC_{1,n-2} + (n-1)NC_{0,n-1} + IC_{1,n-3} + (n-2)RC_{0,n-2}}_{\text{C2D-3}}]y^n \\ &+ [\underbrace{(A-\lambda)C_{1,0} + JC_{0,1}}_{\text{C2D-4}}]x + [\underbrace{(A+K-\lambda)C_{1,1} + 2JC_{0,2} + 2BC_{2,0} + DC_{1,0} + MC_{0,1}}_{\text{C2D-5}}]xy \\ &+ [\underbrace{(A+2K-\lambda)C_{1,2} + 3JC_{0,3} + 2BC_{2,1} + (D+N)C_{1,1} + 2EC_{2,0} + 2MC_{0,2} + HC_{1,0} + QC_{0,1}}_{\text{C2D-6}}]xy^2 \\ &+ \sum_{n=3}^{\infty}[\underbrace{(A+nK-\lambda)C_{1,n} + (n+1)JC_{0,n-1} + 2BC_{2,n-1} + (D+(n-1)N)C_{1,n-1}}_{\text{C2D-7}}\end{aligned}$$

$$
\begin{aligned}
&+ \underbrace{nMC_{0,n} + 2EC_{2,n-2} + (H+(n-2)R)C_{1,n-2} + (n-1)QC_{0,n-1} + 2IC_{2,n-3}}]xy^n \\
&+ [\underbrace{(2A-\lambda)C_{2,0} + JC_{1,1} + CC_{1,0} + LC_{0,1}}_{\text{C2D-8}}]x^2 \\
&+ [\underbrace{(2A+K-\lambda)C_{2,1} + 2JC_{1,2} + 3BC_{3,0} + (C+M)C_{1,1} + 2LC_{0,2} + 2DC_{2,0} + GC_{1,0} + PC_{0,1}}_{\text{C2D-9}}]x^2y \\
&+ [\underbrace{(2A+2K-\lambda)C_{2,2} + 3JC_{1,3} + 3BC_{3,1} + (C+2M)C_{1,2} + 3LC_{0,3} + (2D+N)C_{2,1}}_{\text{C2D-10}} \\
&+ \underbrace{3EC_{3,0} + GC_{1,1} + 2PC_{0,2} + 2HC_{2,0} + QC_{1,1}}]x^2y^2 \\
&+ \sum_{n=3}^{\infty}[\underbrace{(2A+nK-\lambda)C_{2,n} + (n+1)JC_{1,n+1} + 3BC_{3,n-1} + (C+nM)C_{1,n} + (n+1)LC_{0,n+1}}_{\text{C2D-11}} \\
&+ \underbrace{(2D+(n-1)N)C_{2,n-1} + 3EC_{3,m-2} + (G+(n-1)Q)C_{1,n-1} + nPC_{0,n}} \\
&+ \underbrace{+(2H+(n-2)R)C_{2,n-2} + 3IC_{3,n-3}}]x^2y^n \\
&+ \sum_{m=3}^{\infty}[\underbrace{(mA-\lambda)C_{m,0} + JC_{m-1,1} + (m-1)CC_{m-1,0} + LC_{m-2,1} + (m-2)FC_{m-2,0} + OC_{m-3,1}}_{\text{C2D-12}}]x^m \\
&+ \sum_{m=3}^{\infty}[\underbrace{(mA+K-\lambda)C_{m,1} + 2JC_{m-1,2} + (m+1)BC_{m+1,0} + ((m-1)C+M)C_{m-1,1} + 2LC_{m-2,2}}_{\text{C2D-13}} \\
&+ \underbrace{mDC_{m,0} + ((m-2)F+P)C_{m-2,1} + 2OC_{m-3,2} + (m-1)GC_{m-1,0}}]x^my \\
&+ \sum_{m=3}^{\infty}[\underbrace{(mA+2K-\lambda)C_{m,2} + 3JC_{m-1,3} + (m+1)BC_{m+1,1} + ((m-1)C+M)C_{m-1,1}}_{\text{C2D-14}} \\
&+ \underbrace{3LC_{m-2,3} + (n+1)EC_{m+1,0} + ((m-2)F+2P)C_{m-2,2} + ((m-1)G+Q)C_{m-1,1}} \\
&+ \underbrace{3OC_{m-3,3} + mHC_{m,0}}]x^my^2 \\
&+ \sum_{m=3,n=3}^{\infty}[\underbrace{(mA+nK-\lambda)C_{m,n} + (n+1)JC_{m-1,n+1} + (m+1)BC_{m+1,n-1}} \\
&+ \underbrace{((m-1)C+nM)C_{m-1,n} + (n+1)LC_{m-2,n+1} + (mD+(n-1)N)C_{m,n-1} + (m+1)EC_{m+1,n-2}}_{\text{C2D-15}} \\
&+ \underbrace{((m-2)F+nP)C_{m-2,n} + (n+1)OC_{m-3,n+1} + ((m-1)G+(n-1)Q)C_{m-1,n-1}} \\
&+ \underbrace{(mH+(n-2)R)C_{m,n-2} + (m+1)IC_{m+1,n-3}}]x^my^n.
\end{aligned}
$$

Since we are already familiar with construction of "lattice diagrams" from the above equations (C2D-1) to (C2D-15), we omit this step, and proceed directly to the matrix equations.

3.1 Solutions in equivalent matrix form

We now rewrite equations (C2D-1) to (C2D-15) into a sequence of matrix equations for futher study.

$s = 1$:

$$P_1 \boldsymbol{v}_1 = \mathbf{0} \iff$$

$$\begin{bmatrix} A-\lambda & J \\ B & K-\lambda \end{bmatrix} \begin{bmatrix} C_{1,0} \\ C_{0,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$s = 2$:

$$P_2 \boldsymbol{v}_2 + Q_1 \boldsymbol{v}_1 = \boldsymbol{0} \iff$$

$$\begin{bmatrix} 2A-\lambda & J & 0 \\ 2B & A+K-\lambda & 2J \\ 0 & B & 2K-\lambda \end{bmatrix} \begin{bmatrix} C_{2,0} \\ C_{1,1} \\ C_{0,2} \end{bmatrix} + \begin{bmatrix} C & L \\ D & M \\ E & N \end{bmatrix} \begin{bmatrix} C_{1,0} \\ C_{0,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$s = 3$:

$$P_3 \boldsymbol{v}_3 + Q_2 \boldsymbol{v}_2 + R_1 \boldsymbol{v}_1 = \boldsymbol{0} \iff$$

$$\begin{bmatrix} 3A-\lambda & J & 0 & 0 \\ 3B & 2A+K-\lambda & 2J & 0 \\ 0 & 2B & A+2K-\lambda & 3J \\ 0 & 0 & B & 3K-\lambda \end{bmatrix} \begin{bmatrix} C_{3,0} \\ C_{2,1} \\ C_{1,2} \\ C_{0,3} \end{bmatrix} + \begin{bmatrix} 2C & L & 0 \\ 2D & C+M & 2L \\ 2E & D+N & 2M \\ 0 & E & 2N \end{bmatrix} \begin{bmatrix} C_{2,0} \\ C_{1,1} \\ C_{0,2} \end{bmatrix}$$

$$+ \begin{bmatrix} F & O \\ G & P \\ H & Q \\ I & R \end{bmatrix} \begin{bmatrix} C_{1,0} \\ C_{0,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$s = 4$:

$$P_4 \boldsymbol{v}_4 + Q_3 \boldsymbol{v}_3 + R_2 \boldsymbol{v}_2 = \boldsymbol{0} \iff$$

$$\begin{bmatrix} 4A-\lambda & J & 0 & 0 & 0 \\ 4B & 3A+K-\lambda & 2J & 0 & 0 \\ 0 & 3B & 2A+2K-\lambda & 3J & 0 \\ 0 & 0 & 2B & A+3K-\lambda & 4J \\ 0 & 0 & 0 & B & 4K-\lambda \end{bmatrix} \begin{bmatrix} C_{4,0} \\ C_{3,1} \\ C_{2,2} \\ C_{1,3} \\ C_{0,4} \end{bmatrix}$$

$$+ \begin{bmatrix} 3C & L & 0 & 0 \\ 3D & 2C+M & 2L & 0 \\ 3E & 2D+N & C+2M & 3L \\ 0 & 2E & D+2N & 3M \\ 0 & 0 & E & 3N \end{bmatrix} \begin{bmatrix} C_{3,0} \\ C_{2,1} \\ C_{1,2} \\ C_{0,3} \end{bmatrix} + \begin{bmatrix} 2F & O & 0 \\ 2G & F+P & 2O \\ 2H & G+Q & 2P \\ 2I & H+R & 2Q \\ 0 & I & 2R \end{bmatrix} \begin{bmatrix} C_{2,0} \\ C_{1,1} \\ C_{0,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$s = 5$:

$$P_5 \boldsymbol{v}_5 + Q_4 \boldsymbol{v}_4 + R_3 \boldsymbol{v}_3 = \boldsymbol{0} \iff$$

$$\begin{bmatrix} 5A-\lambda & J & 0 & 0 & 0 & 0 \\ 5B & 4A+K-\lambda & 2J & 0 & 0 & 0 \\ 0 & 4B & 3A+2K-\lambda & 3J & 0 & 0 \\ 0 & 0 & 3B & 2A+3K-\lambda & 4J & 0 \\ 0 & 0 & 0 & 2B & A+4K-\lambda & 5J \\ 0 & 0 & 0 & 0 & B & 5K-\lambda \end{bmatrix} \begin{bmatrix} C_{5,0} \\ C_{4,1} \\ C_{3,2} \\ C_{2,3} \\ C_{1,4} \\ C_{0,5} \end{bmatrix}$$

$$+ \begin{bmatrix} 4C & L & 0 & 0 & 0 \\ 4D & 3C+M & 2L & 0 & 0 \\ 4E & 3D+N & 2C+2M & 3L & 0 \\ 0 & 3E & 2D+2N & C+3M & 4L \\ 0 & 0 & 2E & D+3N & 4M \\ 0 & 0 & 0 & E & 4N \end{bmatrix} \begin{bmatrix} C_{4,0} \\ C_{3,1} \\ C_{2,2} \\ C_{1,3} \\ C_{0,4} \end{bmatrix} + \begin{bmatrix} 3F & O & 0 & 0 \\ 3G & 2F+P & 2O & 0 \\ 3H & 2G+Q & F+2P & 3O \\ 3I & 2H+R & G+2Q & 3P \\ 0 & 2I & H+2R & 3Q \\ 0 & 0 & I & 3R \end{bmatrix} \begin{bmatrix} C_{3,0} \\ C_{2,1} \\ C_{1,2} \\ C_{0,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\vdots$

For arbitrary s:

$$P_s\boldsymbol{v}_s + Q_{s-1}\boldsymbol{v}_{s-1} + R_{s-2}\boldsymbol{v}_{s-2} = \mathbf{0} \iff$$

$$\begin{bmatrix} sA-\lambda & J & 0 & \dots & & \\ sB & (s-1)A+K-\lambda & 2J & 0 & \dots & \\ 0 & (s-1)A & (s-2)A+2K-\lambda & 3J & 0 & \dots \\ & \ddots & \ddots & \ddots & & \\ \dots & \dots & 0 & 2B & A+(s-1)K-\lambda & sJ \\ \dots & \dots & \dots & 0 & B & sK-\lambda \end{bmatrix} \begin{bmatrix} C_{s,0} \\ C_{s-1,1} \\ C_{s-2,2} \\ \vdots \\ C_{1,s-1} \\ C_{0,s} \end{bmatrix}$$

$$+ \begin{bmatrix} (s-1)C & L & 0 & \dots & & \\ (s-1)D & (s-2)C+M & 2L & 0 & \dots & \\ (s-1)E & (s-2)D+N & (s-3)C+2M & 3L & 0 & \dots \\ 0 & (s-2)E & (s-3)D+2N & (s-4)C+3M & 4L & \\ & \ddots & \ddots & \ddots & & \\ 0 & \dots & & 2E & D+(s-2)N & (s-1)M \\ 0 & \dots & & & E & (s-1)N \end{bmatrix} \begin{bmatrix} C_{s-1,0} \\ C_{s-2,1} \\ \vdots \\ C_{1,s-2} \\ C_{0,s-1} \end{bmatrix}$$

$$+ \begin{bmatrix} (s-2)F & O & \dots & & & \\ (s-2)G & (s-3)F+P & 2O & & & \\ (s-2)H & (s-3)G+Q & (s-4)F+2P & 3O & & \\ (s-2)I & (s-3)H+R & (s-4)G+2Q & (s-5)F+3P & 4O & \\ & \ddots & \ddots & \ddots & & \\ 0 & \dots & & & & (s-2)O \\ 0 & \dots & & & & (s-2)P \\ 0 & \dots & & & & (s-2)Q \\ 0 & \dots & & & I & (s-2)R \end{bmatrix} \begin{bmatrix} C_{s-2,0} \\ C_{s-3,1} \\ \\ \vdots \\ \\ C_{1,s-3} \\ C_{0,s-2} \end{bmatrix}$$

If we are interested in a conserved quantity of power $n \geq 3$, then the following relations must hold, where λ is an eigenvalue of matrix P_k for some $k \in \{1, 2, \dots, n\}$.

$$\begin{aligned} P_1(\lambda)\boldsymbol{v}_1 &= \mathbf{0} \\ P_2(\lambda)\boldsymbol{v}_2 + Q_1\boldsymbol{v}_1 &= \mathbf{0} \\ P_3(\lambda)\boldsymbol{v}_3 + Q_2\boldsymbol{v}_2 + R_1\boldsymbol{v}_1 &= \mathbf{0} \\ &\vdots \\ P_n(\lambda)\boldsymbol{v}_n + Q_{n-1}\boldsymbol{v}_{n-1} + R_{n-2}\boldsymbol{v}_{n-2} &= \mathbf{0} \\ Q_n\boldsymbol{v}_n + R_{n-1}\boldsymbol{v}_{n-1} &= \mathbf{0} \\ R_n\boldsymbol{v}_n &= \mathbf{0} \end{aligned}$$

3.2 Algorithm

In order to solve the vectors $\boldsymbol{v}_i$ for $i = 1, 2, \dots$, up to the order of our interest, we introduce the following algorithm.

Algorithm 3 Conserved quantity of order d

```
Given matrices P_1, P_2, ... and Q_1, Q_2, ...:
  cq ← list of conserved quantities
  for k = 1, ..., d do
    v ← {0}_{n=1}^{d}
    eigs ← {(λ_i^(k), u_i^(k))}_{i=1}^{k+1} (eigen-pairs of P_k)
    for i = 1, ..., k+1 do
      solExist ← True
      v[k] ← u_i^(k)
      if k + 1 ≤ d then
        if P_{k+1}(λ_i^(k)) x + Q_k v[k] = 0 has No solution then
          solExist ← False
        else
          v[k+1] ← solving P_{k+1}(λ_i^(k)) x + Q_k v[k] = 0 for x
        end if
      end if
      for j = k + 2, ..., d and solExist do
        if P_j(λ_j^(k)) x + Q_{j-1} v[j-1] + R_{j-2} v[j-2] = 0 has No solution then
          solExist ← False
        else
          v[j] ← solving P_j(λ_j^(k)) x + Q_{j-1} v[j-1] + R_{j-2} v[j-2] = 0 for x
        end if
      end for
      if solExist and Q_d v[d] + R_{d-1} v[d-1] = 0 and R_d v[d] = 0 then
        cq.insert(v)
      end if
    end for
  end for
```

3.3 Examples

In this section, we will test our algorithm on a few examples. The first one is 2D Hamiltonian system given by

$$\dot{x} = Ax + By + Cx^2 - 2Jxy + Ey^2,$$
$$\dot{y} = Fx - Ay + Hx^2 - 2Cxy + Jy^2.$$

The Hamiltonian for this system is

$$H(x,y) = \frac{F}{2}x^2 - Axy - \frac{B}{2}y^2 + \frac{H}{3}x^3 - Cx^2y + Jxy^2 - \frac{E}{3}y^3 \tag{3.5}$$

In order to follow the steps in our algorithm, we need to find the matrices $P_1, P_2, \ldots, Q_1, Q_2, \ldots$ and $R_1, R_2, \ldots$. Since it is a 2D system, we have $0 = R_1 = R_2 = \ldots$.

$$P_1 = \begin{bmatrix} A & F \\ B & -A \end{bmatrix}, P_2 = \begin{bmatrix} 2A & F & 0 \\ 2B & 0 & 2F \\ 0 & B & -2A \end{bmatrix}, P_3 = \begin{bmatrix} 3A & F & 0 & 0 \\ 3B & A & 2F & 0 \\ 0 & 2B & -A & 3F \\ 0 & 0 & B & -3A \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} C & H \\ -2J & -2C \\ E & J \end{bmatrix}, Q_2 = \begin{bmatrix} 2C & H & 0 \\ -4J & -C & 2H \\ 2E & -J & -4C \\ 0 & E & 2J \end{bmatrix}, Q_3 = \begin{bmatrix} 3C & H & 0 & 0 \\ -6J & 0 & 2H & 0 \\ 3E & -3J & -3C & 3H \\ 0 & 2E & 0 & -6C \\ 0 & 0 & E & 3J \end{bmatrix}$$

Next, we will first show the steps to detive the Hamiltonian using our algorithm. We start with $k = 2$ for saving time, then we need to solve

$$P_2(\lambda)\boldsymbol{v}_2 = 0 \text{ for some undetermined } \lambda.$$

Three eigen-pairs are

$$\begin{aligned} \lambda_1^{(2)} &= 0, \boldsymbol{u}_1^{(2)} = \begin{bmatrix} -F & 2A & B \end{bmatrix}^T \\ \lambda_2^{(2)} &= 2\sqrt{A^2+BF}, \boldsymbol{u}_2^{(2)} = \begin{bmatrix} (A+\sqrt{A^2+BF})F & 2BF & (-A+\sqrt{A^2+BF})B \end{bmatrix}^T \\ \lambda_3^{(2)} &= -2\sqrt{A^2+BF}, \boldsymbol{u}_3^{(2)} = \begin{bmatrix} (A-\sqrt{A^2+BF})F & 2BF & (-A-\sqrt{A^2+BF})B \end{bmatrix}^T. \end{aligned}$$

We continue with $\lambda_1^{(2)} = 0$, $\boldsymbol{u}_1^{(2)} = \begin{bmatrix} -F & 2A & B \end{bmatrix}^T$. Set $\boldsymbol{v}_1 = \boldsymbol{0}$ and $\boldsymbol{v}_2 = \boldsymbol{u}_1^{(2)}$, solve

$$P_3(0)\boldsymbol{v}_3 + Q_2\boldsymbol{v}_2 = 0$$

$$\begin{bmatrix} 3A & F & 0 & 0 \\ 3B & A & 2F & 0 \\ 0 & 2B & -A & 3F \\ 0 & 0 & B & -3A \end{bmatrix} \boldsymbol{v}_3 = \begin{bmatrix} -2C & -H & 0 \\ 4J & C & -2H \\ -2E & J & 4C \\ 0 & -E & -2J \end{bmatrix} \begin{bmatrix} -F \\ 2A \\ B \end{bmatrix}$$

to obtain $\boldsymbol{v}_3 = \begin{bmatrix} -\frac{2}{3}H & 2C & -2J & \frac{2}{3}E \end{bmatrix}^T$.
Finally, we see that

$$Q_3\boldsymbol{v}_3 = \begin{bmatrix} 3C & H & 0 & 0 \\ -6J & 0 & 2H & 0 \\ 3E & -3J & -3C & 3H \\ 0 & 2E & 0 & -6C \\ 0 & 0 & E & 3J \end{bmatrix} \begin{bmatrix} -\frac{2}{3}H \\ 2C \\ -2J \\ \frac{2}{3}E \end{bmatrix} = \boldsymbol{0}.$$

From $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$, the derived Hamiltonian is given by

$$H(x,y) = -Fx^2 + 2Axy + By^2 - \frac{2}{3}Hx^3 + 2Cx^2y - 2Jxy^2 + \frac{2}{3}Ey^2,$$

which is equivalent to equation(3.5).
However, if we choose $\lambda_2^{(2)} = 2\sqrt{A^2+BF}$, $\boldsymbol{u}_2^{(2)} = \begin{bmatrix} (A+\sqrt{A^2+BF})F & 2BF & (-A+\sqrt{A^2+BF})B \end{bmatrix}^T$, it will not lead a conserved quantity, as we will show below.
Notice that $\boldsymbol{v}_3 = \begin{bmatrix} H & -3C & 3J & -E \end{bmatrix}^T$ is the only solution to $Q_3\boldsymbol{v}_3 = 0$. However, there is no solution to $\boldsymbol{v}_2$ that satisfies

$$P_3(\lambda_2)\boldsymbol{v}_2 = -Q_2\boldsymbol{v}_2$$

$$\begin{bmatrix} 3A-\lambda_2 & F & 0 & 0 \\ 3B & A-\lambda_2 & 2F & 0 \\ 0 & 2B & -A-\lambda_2 & 3F \\ 0 & 0 & B & -3A-\lambda_2 \end{bmatrix} \begin{bmatrix} H \\ -3C \\ 3J \\ -E \end{bmatrix} = \begin{bmatrix} -2C & -H & 0 \\ 4J & C & -2H \\ -2E & J & 4C \\ 0 & -E & -2J \end{bmatrix} \boldsymbol{v}_2.$$

Hence, our choice of λ_2 does not lead to a conserved quantity. Similar argument holds for λ_3. So the derived Hamiltonian is the only conserved quantity of order 3.

Next, we will see the example of 3D-Hamiltonian system:

$$\dot{x} = Ax + By + Cx^2 - 2Nxy + Ey^2 + Fx^3 + Gx^2y - 3Rxy^2 + Iy^3$$
$$\dot{y} = Jx - Ay + Lx^2 - 2Cxy + Ny^2 + Ox^3 - 3Fx^2y - Gxy^2 + Ry^3.$$

The Hamiltonian for the system is given by

$$\mathcal{H}(x,y) = \frac{J}{2}x^2 - Axy - \frac{B}{2}y^2 + \frac{L}{3}x^3 - Cx^2y + Nxy^2 - \frac{E}{3}y^3 + \frac{O}{4}x^4 - Fx^3y - \frac{G}{2}x^2y^2 + Rxy^3 - \frac{I}{4}y^4. \tag{3.6}$$

In order to derive such conserved quantity, we need to show the following:

$$\begin{aligned} P_2(\lambda)\boldsymbol{v}_2 &= 0 \\ P_3(\lambda)\boldsymbol{v}_3 + Q_2\boldsymbol{v}_2 &= 0 \\ P_4(\lambda)\boldsymbol{v}_4 + Q_3\boldsymbol{v}_3 + R_2\boldsymbol{v}_2 &= 0 \\ Q_4\boldsymbol{v}_4 + R_3\boldsymbol{v}_3 &= 0 \\ R_4\boldsymbol{v}_4 &= 0. \end{aligned}$$

Step 1: Let $k = 2$, the eigenvalues and eigenvectors of $P_2 = \begin{bmatrix} 2A & J & 0 \\ 2B & 0 & 2J \\ 0 & B & -2A \end{bmatrix}$ are

$$\begin{aligned} \lambda_1^{(2)} &= 0, \boldsymbol{u}_1^{(2)} = \begin{bmatrix} -J & 2A & B \end{bmatrix}^T \\ \lambda_2^{(2)} &= 2\sqrt{A^2+BJ}, \boldsymbol{u}_2^{(2)} = \begin{bmatrix} (A+\sqrt{A^2+BJ})J & 2BJ & (-A+\sqrt{A^2+BJ})B \end{bmatrix}^T \\ \lambda_3^{(2)} &= -2\sqrt{A^2+BJ}, \boldsymbol{u}_3^{(2)} = \begin{bmatrix} (A-\sqrt{A^2+BJ})J & 2BJ & (-A-\sqrt{A^2+BJ})B \end{bmatrix}^T. \end{aligned}$$

We choose eigenvalue $\lambda = \lambda_1^{(2)} = 0$ and $\boldsymbol{v}_2 = \begin{bmatrix} -J & 2A & B \end{bmatrix}^T$.

Step 2: Solve

$$P_3(0)\boldsymbol{v}_3 = -Q_2\boldsymbol{v}_2$$
$$\begin{bmatrix} 3A & J & 0 & 0 \\ 3B & A & 2J & 0 \\ 0 & 2B & -A & 3J \\ 0 & 0 & B & -3A \end{bmatrix} \boldsymbol{v}_3 = \begin{bmatrix} -2C & -L & 0 \\ 4N & C & -2L \\ -2E & N & 4C \\ 0 & -E & -2N \end{bmatrix} \begin{bmatrix} -J \\ 2A \\ B \end{bmatrix}$$

to obtain $\boldsymbol{v}_3 = \begin{bmatrix} -\frac{2}{3}L & 2C & -2N & \frac{2}{3}E \end{bmatrix}^T$.

Step 3: Solving

$$P_4(0)\boldsymbol{v}_4 = -Q_3\boldsymbol{v}_3 - R_2\boldsymbol{v}_2$$
$$\begin{bmatrix} 4A & J & 0 & 0 & 0 \\ 4B & 2A & 2J & 0 & 0 \\ 0 & 3B & 0 & 3J & 0 \\ 0 & 0 & 2B & -2A & 4J \\ 0 & 0 & 0 & B & -4A \end{bmatrix} \boldsymbol{v}_4 = -\begin{bmatrix} 3C & L & 0 & 0 \\ -6N & 0 & 2L & 0 \\ 3E & -3N & -3C & 3L \\ 0 & 2E & 0 & -6C \\ 0 & 0 & E & 3N \end{bmatrix} \begin{bmatrix} -\frac{2}{3}L \\ 2C \\ -2N \\ \frac{2}{3}E \end{bmatrix} - \begin{bmatrix} 2F & O & 0 \\ 2G & -2F & 2O \\ -6R & 0 & -6F \\ 2I & -2R & -2G \\ 0 & I & 2R \end{bmatrix} \begin{bmatrix} -J \\ 2A \\ B \end{bmatrix}$$

gives $\begin{bmatrix} -\frac{O}{2} & 2F & G & -2R & \frac{I}{2} \end{bmatrix}^T$.

Step 4:

$$Q_4\boldsymbol{v}_4+R_3\boldsymbol{v}_3 = \begin{bmatrix} 4C & L & 0 & 0 & 0 \\ -8N & C & 2L & 0 & 0 \\ 4E & -5N & -2C & 3L & 0 \\ 0 & 3E & -2N & -5C & 4L \\ 0 & 0 & 2E & N & -8C \\ 0 & 0 & 0 & E & 4N \end{bmatrix} \begin{bmatrix} -\frac{O}{2} \\ 2F \\ G \\ -2R \\ \frac{I}{2} \end{bmatrix} + \begin{bmatrix} 3F & O & 0 & 0 \\ 3G & -F & 2O & 0 \\ -9R & G & -5F & 3O \\ 3I & -5R & -G & -9F \\ 0 & 2I & -R & -3G \\ 0 & 0 & I & 3R \end{bmatrix} \begin{bmatrix} -\frac{2}{3}L \\ 2C \\ -2N \\ \frac{2}{3}E \end{bmatrix} = \mathbf{0}$$

Step 5:

$$R_4\boldsymbol{v}_4 = \begin{bmatrix} 4F & O & 0 & 0 & 0 \\ 4G & 0 & 2O & 0 & 0 \\ -12R & 2G & -4F & 3O & 0 \\ 4I & -8R & 0 & -8F & 4O \\ 0 & 3I & -4R & -2G & -12F \\ 0 & 0 & 2I & 0 & -4G \\ 0 & 0 & 0 & I & 4R \end{bmatrix} \begin{bmatrix} -\frac{O}{2} \\ 2F \\ G \\ -2R \\ \frac{I}{2} \end{bmatrix} = \mathbf{0}.$$

It follows from the above five steps that the Hamiltonian is derived as

$$\mathcal{H}(x,y) = -Jx^2+2Axy+By^2-\frac{2}{3}Lx^3+2Cx^2y-2Nxy^2+\frac{2}{3}Ey^3-\frac{O}{2}x^4+2Fx^3y+Gx^2y^2-2Rxy^3+\frac{I}{2}y^4,$$

which is equivalent to equation (3.6).

Chapter 4

3D System: Lorenz System

Lorenz system is a well studied dynamical system in $\mathbb{R}^3$ due to its connection to chaotic behavior, which has the form,

$$\begin{aligned} \dot{x} &= s(y-x), \\ \dot{y} &= rx - xz - y, \\ \dot{z} &= xy - bz, \end{aligned} \tag{L3D}$$

where s, r and b are nonnegative real parameters. Parameters play an important role in the behaviors of dynamical systems, especially when we are trying to find conserved quantities based an arbitrary parameters. As we shall see later, assuming conserved quantity of a certain form will restrain the possible values or ranges of these parameters. In this section, we are interested in analytic conserved quantities of system (L3D), which has the form

$$\psi(x, y, z, t) = f(x, y, z)e^{-kt} \tag{4.1}$$

for some $k \in \mathbb{R}$. This class of conserved quantity was studied in [3] and it was shown that there are six nontrivial polynomial conserved quantities [2][1]. In order to derive a formula for $f(x, y, z)$, we hypothesize that it is analytic around the origin given by

$$f(x, y, z) = \sum_{l=0,m=0,n=0}^{\infty} C_{l,m,n}\, x^l y^m z^n, \tag{4.2}$$

where $C_{l,m,n}$ is underdetermined real coefficients. In order to follow the similar approach as in the Q2D system, we need to compute $\frac{\mathrm{d}f}{\mathrm{d}t}$ by the chain rule. We start the process by computing the partial derivatives of $f(x, y, z)$

$$\begin{aligned} f_x &= \sum_{l=0,m=0,n=0}^{\infty} (l+1)C_{l+1,m,n}\, x^l y^m z^n, \\ f_y &= \sum_{l=0,m=0,n=0}^{\infty} (m+1)C_{l,m+1,n}\, x^l y^m z^n, \\ f_z &= \sum_{l=0,m=0,n=0}^{\infty} (n+1)C_{l,m,n+1}\, x^l y^m z^n. \end{aligned} \tag{4.3}$$

By the definition of conserved quantity,

$$0 = \frac{\mathrm{d}\psi}{\mathrm{d}t} = \left(\frac{\mathrm{d}f}{\mathrm{d}t} - kf\right)e^{-kt} = f_x\dot{x} + f_y\dot{y} + f_z\dot{z} - kf.$$

Substituting in (4.3) and system (L3D) leads to

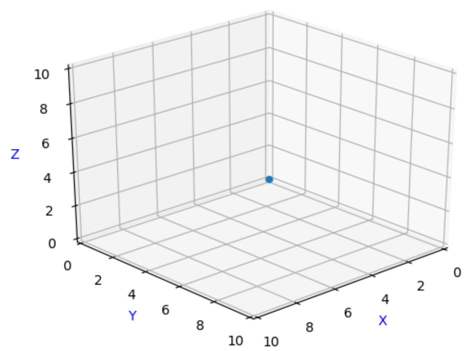
$$
\begin{aligned}
0 = & \sum_{l=0,m=0,n=0}^{\infty} \{[s(l+1)C_{l+1,m,n} - (m+1)C_{l,m+1,n}]x^l y^{m+1} z^n + [r(m+1)C_{l,m+1,n} \\
& - s(l+1)C_{l+1,m,n}]x^{l+1}y^m z^n + [-b(n+1)C_{l,m,n+1}]x^l y^m z^{n+1} + (n+1)C_{l,m,n+1}\, x^{l+1}y^{m+1}z^n \\
& + [-(m+1)C_{l,m+1,n}]x^{l+1}y^m z^{n+1} - kC_{l,m,n}\, x^l y^m z^n\} \\
= & \underbrace{-kC_{0,0,0}}_{\text{L3D-1}} + \sum_{l=1}^{\infty} \underbrace{[rC_{l-1,1,0} - (sl+k)C_{l,0,0}]}_{\text{L3D-2}} x^l + \sum_{m=1}^{\infty} \underbrace{[sC_{1,m-1,0} - (m+k)C_{0,m,0}]}_{\text{L3D-3}} y^m \\
& + \sum_{n=1}^{\infty} \underbrace{[-(b+k)C_{0,0,n}]}_{\text{L3D-4}} z^n + \sum_{l=1,m=1}^{\infty} \underbrace{[s(l+1)C_{l+1,m-1,0} + r(m+1)C_{l-1,m+1,0} - (m+sl+k)C_{l,m,0}}_{\text{L3D-5}} \\
& \underbrace{+C_{l-1,m-1,1}]x^l y^m}\, x^l y^m + \sum_{l=1,n=1}^{\infty} \underbrace{[rC_{l-1,1,n} - (sl+bn+k)C_{l,0,n} - C_{l-1,1,n-1}]}_{\text{L3D-6}} x^l z^n \\
& + \sum_{m=1,n=1}^{\infty} \underbrace{[sC_{1,m-1,n} - (m+bn+k)C_{0,m,n}]}_{\text{L3D-7}} y^m z^n + \sum_{l=1,m=1,n=1}^{\infty} \underbrace{[s(l+1)C_{l+1,m-1,n} - (m+sl} \\
& \underbrace{+bn+k)C_{l,m,n} + r(m+1)C_{l-1,m+1,n} + (n+1)C_{l-1,m-1,n+1} - (m+1)C_{l-1,m+1,n-1}]}_{\text{L3D-8}} x^l y^m z^n
\end{aligned}
$$

4.1 Equations and lattices

Again as in Q2D system, we will study the relations between coefficients $C_{l,m,n}$ in equation (4.2) by lattice diagrams introduced previously. The lattices in the same color have the same relation in terms of x and y, varying in n. For example, (L3D-2) and (L3D-6), (L3D-3) and (L3D-7) and (L3D-4) and (L3D-8) share the same patterns, respectively.

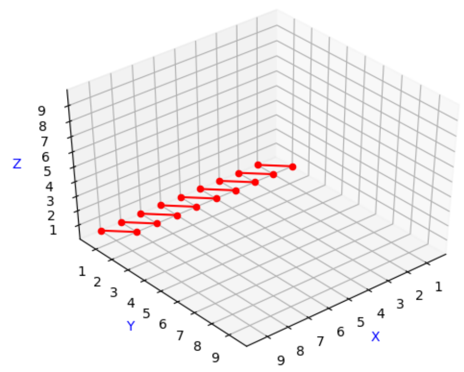
(L3D-1)

$$kC_{0,0,0} = 0$$



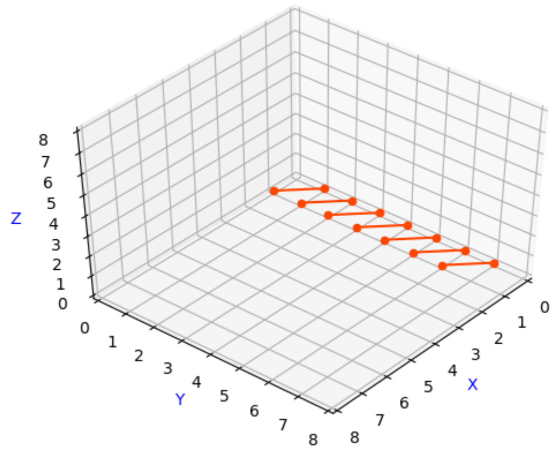
(L3D-2)

$$rC_{l-1,1,0} - (sl+k)C_{l,0,0} = 0,\ l \geq 0$$



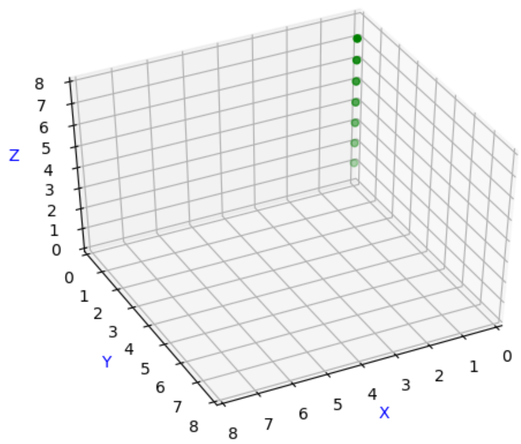
(L3D-3)

$$sC_{1,m-1,0} - (k+m)C_{0,m,0} = 0,\ m \geq 1$$



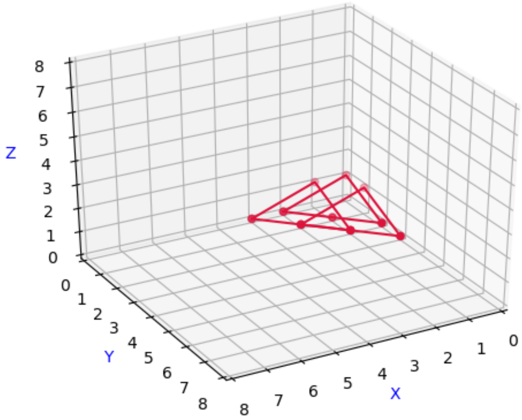
(L3D-4)

$$(k+bn)C_{0,0,n} = 0,\ n \geq 1$$



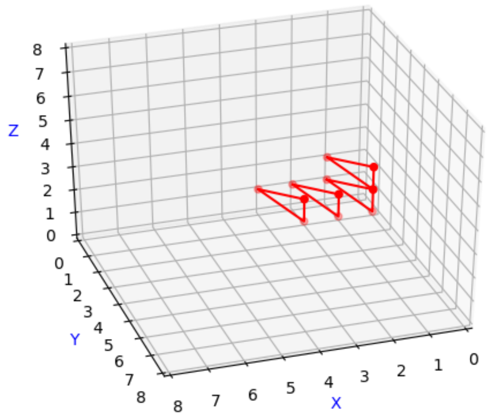
(L3D-5)

$$s(l+1)C_{l+1,m-1,0} + r(m+1)C_{l-1,m+1,0} - (m+sl+k)C_{l,m,0} + C_{l-1,m-1,1} = 0,\ l,m \geq 1$$



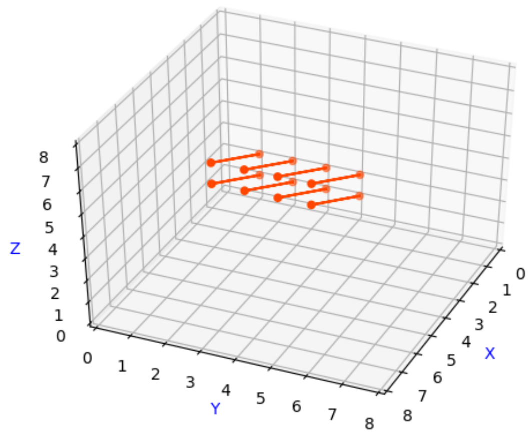
(L3D-6)

$$rC_{l-1,1,n} - (sl + bn + k)C_{l,0,n} - C_{l-1,1,n-1} = 0,\ l, n \geq 1$$



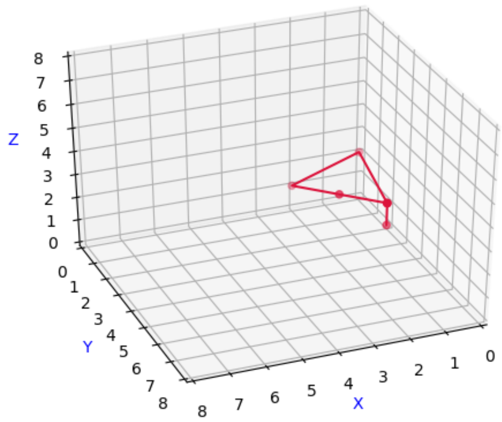
(L3D-7)

$$sC_{1,m-1,n} - (m + bn + k)C_{0,m,n} = 0,\ m, n \geq 1$$



(L3D-8)

$$s(l+1)C_{l+1,m-1,n} - (m + sl + bn + k)C_{l,m,n} + r(m+1)C_{l-1,m+1,n}$$
$$+(n + 1C_{l-1,m-1,n+1} - (m+1)C_{l-1,m+1,n-1} = 0,\ l, m, n \geq 1$$



4.2 Solutions in equivalent matrix form

Again as in the (Q2D), for the purpose of computation, it will be helpful to group the equations into matrix forms using the lattice equations.
Let M be the sum of powers of x and y, n be the power of z.
$M = 0$:

$$\underbrace{(k+bn)}_{P_{0,n}}\underbrace{C_{0,0,n}}_{\boldsymbol{v}_{0,n}} = 0,\ n \geq 0. \tag{4.4}$$

$M = 1$:

$$n = 0: \underbrace{\begin{bmatrix} -(s+k) & r \\ s & -(1+k) \end{bmatrix}}_{P_{1,0}}\underbrace{\begin{bmatrix} C_{1,0,0} \\ C_{0,1,0} \end{bmatrix}}_{\boldsymbol{v}_{1,0}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.5}$$

$$n \geq 1: \underbrace{\begin{bmatrix} -(s+k+bn) & r \\ s & -(1+k+bn) \end{bmatrix}}_{P_{1,n}}\underbrace{\begin{bmatrix} C_{1,0,n} \\ C_{0,1,n} \end{bmatrix}}_{\boldsymbol{v}_{1,n}} = \begin{bmatrix} C_{0,1,n-1} \\ 0 \end{bmatrix}. \tag{4.6}$$

$M = 2$:

$$n = 0: \underbrace{\begin{bmatrix} -(2s+k) & r & 0 \\ 2s & -(s+1+k) & 2r \\ 0 & s & -(2+k) \end{bmatrix}}_{P_{2,0}}\underbrace{\begin{bmatrix} C_{2,0,0} \\ C_{1,1,0} \\ C_{0,1,0} \end{bmatrix}}_{\boldsymbol{v}_{2,0}} = \begin{bmatrix} 0 \\ -C_{0,0,1} \\ 0 \end{bmatrix}. \tag{4.7}$$

$n \geq 1$:

$$\underbrace{\begin{bmatrix} -(2s+k+bn) & r & 0 \\ 2s & -(s+1+k+bn) & 2r \\ 0 & s & -(2+k+bn) \end{bmatrix}}_{P_{2,n}}\underbrace{\begin{bmatrix} C_{2,0,n} \\ C_{1,1,n} \\ C_{0,2,n} \end{bmatrix}}_{\boldsymbol{v}_{2,n}} = \begin{bmatrix} C_{1,1,n-1} \\ -(n+1)C_{0,0,n+1} + 2C_{0,2,n-1} \\ 0 \end{bmatrix}. \tag{4.8}$$

$M = m$: $n = 0$:

$$\underbrace{\begin{bmatrix} -(ms+k) & r & 0 & \dots & 0 & & & & \\ ms & -[(m-1)s+1+k] & 2r & 0 & \dots & 0 & & & \\ \ddots & \ddots & \ddots & & & & & & \\ 0 & \dots & 0 & -(m-j)s & -[(m-1-j)s+1+j+k] & (j+2)r & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & & & & & & \\ 0 & \dots & 0 & 2d & -(s+m-1+k) & mr & 0 & \dots & 0 \\ 0 & \dots & 0 & s & -(m+k) & & & & \end{bmatrix}}_{P_{m,0}}$$

$$\underbrace{\begin{bmatrix} C_{m,0,0} \\ C_{m-1,1,0} \\ \vdots \\ C_{m-1-j,j+1,0} \\ \vdots \\ C_{1,m-1,0} \\ C_{0,m,0} \end{bmatrix}}_{\boldsymbol{v}_{m,0}} = \begin{bmatrix} 0 \\ -C_{m-2,0,1} \\ \vdots \\ -C_{m-2-j,j,1} \\ \vdots \\ -C_{0,m-2,1}\,0 \\ 0 \end{bmatrix} \tag{4.9}$$

$n \geq 1$:

$$\underbrace{\begin{bmatrix} -(ms+k+bn) & r & 0 & \dots & 0 & & & & \\ ms & -[(m-1)s+1+k+bn & 2r & 0 & \dots & 0 & & & \\ \ddots & \ddots & \ddots & & & & & & \\ 0 & \dots & 0 & -(m-j)s & -[(m-1-j)s+1+j+k+bn] & (j+2)r & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & & & & & & \\ 0 & \dots & 0 & 2d & -(s+m-1+k+bn) & mr & 0 & \dots & 0 \\ 0 & \dots & 0 & s & -(m+k+bn) & & & & \end{bmatrix}}_{P_{m,n}}$$

$$\underbrace{\begin{bmatrix} C_{m,0,n} \\ C_{m-1,1,n} \\ \vdots \\ C_{m-1-j,j+1,n} \\ \vdots \\ C_{1,m-1,n} \\ C_{0,m,n} \end{bmatrix}}_{\boldsymbol{v}_{m,n}} = \begin{bmatrix} C_{m-1,1,n-1} \\ -(n+1)C_{m-2,0,n+1} + 2C_{m-2,2,n-1} \\ \vdots \\ -(n+1)C_{m-2-j,j,n+1} + (j+2)C_{m-2-j,j+2,n-1} \\ \vdots \\ -(n+1)C_{0,m-2,n+1} + mC_{0,m,n-1} \\ 0 \end{bmatrix} \tag{4.10}$$

Starting from matrix equation (4.4), if for any $k \in \mathbb{R}$, $\det P_{0,n} \neq 0$ for all $n \geq 0$, it will lead to $\boldsymbol{v}_{0,n} = \mathbf{0}$ for all $n \geq 0$. Following from $\boldsymbol{v}_{0,n} = \mathbf{0}$, $n \geq 0$, if we continue assuming $\det P_{2,n} \neq 0$ for all $n \geq 0$, we will again have $\boldsymbol{v}_{2,n} = \mathbf{0}$ for $n \geq 0$. By induction, this will lead to $\boldsymbol{v}_{2l,n} = 0$ for all $l \geq 0, n \geq 0$. Similarly, in matrix equation (4.5) and (4.6), if for any $k \in \mathbb{R}$ $\det P_{1,n} \neq 0$ for all $n \geq 0$, then we will have $\boldsymbol{v}_{1,n} = \mathbf{0}$ for all $n \geq 0$. And moreover, if $\det P_{2l+1,n} \neq 0$ for all $l \geq 0, n \geq 0$, we will have $\boldsymbol{v}_{2l+1,n} = 0$ for all $l, n \geq 0$. This illustrates the following theorem.

Theorem 4. *If for $k \in \mathbb{R}$ $\det P_{m,n} \neq 0$ for all $m, n \geq 0$, then there is no nontrivial analytic conserved quantity of the form $f(x,y,z)e^{-kt}$.*

One way to interprete this is that a conserved quantity of the form (4.1) exists only if $P_{m,n}$ is singular for some m and n. This constraint will restrict the choices of parameters in order to have such conserved quantity. Furthermore, we will investigate special ones which will lead $f(x,y,z)$ to be a polynomial.

In the section B of the appendix, we showed using a tree search algorithm on the lattice equations to verify that all conserved quantities of up to order four polynomial of the Lorenz system (L3D). Specifically, we will recover all the six known conserved quantities.

Chapter 5

A Nondissipative Chaotic System

In this section, we continue applying our lattice approach to the following 3D system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + yz, \\ \dot{z} &= 1 - y^2. \end{aligned} \tag{Q3D}$$

This is a nondissipative chaotic system discovered by [5], which has a simpler form than the dissipative Lorenz system. Again, we hope to find conserved quantities of the form

$$\psi(x, y, z, t) = f(x, y, z)e^{-kt}, \tag{5.1}$$

where $k \in \mathbb{R}$ and $f(x, y, z) : U \times I \to \mathbb{R}$ is analytic around the origin. Assuming conserved quantities in the from (5.1) exist, we will need to solve

$$0 = \frac{\mathrm{d}f}{\mathrm{d}t} - kf \tag{5.2}$$

to obtain the coefficients of the conserved quantity. Recall that we have

$$\begin{aligned} f_x &= \sum_{l=0,m=0,n=0}^{\infty} (l+1)C_{l+1,m,n}\, x^l y^m z^n, \\ f_y &= \sum_{l=0,m=0,n=0}^{\infty} (m+1)C_{m,l+1,n}\, x^l y^m z^n, \\ f_x &= \sum_{l=0,m=0,n=0}^{\infty} (n+1)C_{l,m,n+1}\, x^l y^m z^n. \end{aligned} \tag{5.3}$$

Combining (5.2), (Q3D) and (5.3) and omitting the intermediate steps as illustrated before, we will have

$$\begin{aligned} 0 = &\sum_{l=1,m=2}^{\infty} [\underbrace{(l+1)C_{l+1,m-1,0} - (m+1)C_{l-1,m+1,0} + C_{l,m,1} - kC_{l,m,0} - C_{l,m-2,1}}_{\text{Q3D-1}}]x^l y^m \\ &+ \sum_{n=0}^{\infty}[\underbrace{(n+1)C_{0,0,n+1} - kC_{0,0,n}}_{\text{Q3D-2}}]z^n + [\underbrace{C_{1,0,0} + C_{0,1,0} - kC_{0,1,0}}_{\text{Q3D-3}}]y + \sum_{n=1}^{\infty}[\underbrace{C_{1,0,n} + C_{0,1,n-1}}_{} \\ &\underbrace{+(n+1)C_{0,1,n+1} - kC_{0,1,n}}_{\text{Q3D-4}}]yz^n + \sum_{m=2}^{\infty}[\underbrace{C_{1,m-1,0} + C_{0,m,1} - kC_{0,m,0} - C_{0,m-2,1}}_{\text{Q3D-5}}]y^m \end{aligned}$$

$$+\sum_{m=2,n=1}^{\infty}[\underbrace{C_{1,m-1,n}+mC_{0,m,n+1}+(n+1)C_{0,m,n+1}-kC_{0,1,n+1}-kC_{0,1,n}}_{\text{Q3D-6}}]y^m z^n$$

$$+\sum_{l=1,n=0}^{\infty}[\underbrace{-C_{l-1,1,n}+(n+1)C_{l,0,n+1}-kC_{l,0,n}}_{\text{Q3D-7}}]x^l z^n+\sum_{l=1}^{\infty}[\underbrace{(l+1)C_{l+1,0,0}-2C_{l-1,2,0}+C_{l,1,1}}_{\text{Q3D-8}}$$

$$\underbrace{-kC_{l,1,0}}]x^l y+\sum_{l=1,n=1}^{\infty}[\underbrace{(l+1)C_{l+1,0,n}-2C_{l-1,2,n}+C_{l,1,n-1}+(n+1)C_{l,1,n+1}-kC_{l,1,n}}_{\text{Q3D-9}}]x^l y z^n$$

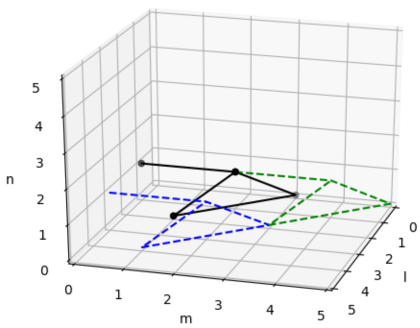
$$+\sum_{l=1,m=2,n=1}^{\infty}[\underbrace{(l+1)C_{l+1,m-1,n}-(m+1)C_{l-1,m+1,n}+mC_{l,m,n-1}+(n+1)C_{l,m,n+1}}$$

$$\underbrace{-kC_{l,m,n}-(n+1)C_{l,m-2,n+1}}_{\text{Q3D-10}}]x^l y^m z^n$$

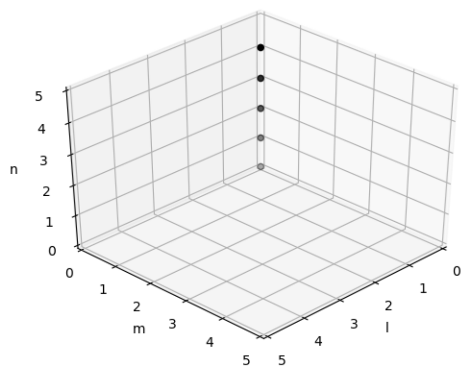
5.1 Equations and lattices

Again, we followed a similar approach as in the Lorenz system. Since it is known that the system (Q3D) is nondissipative, we can assume $k=0$ to simplify our analysis. Then the above computation leads to the following equations.

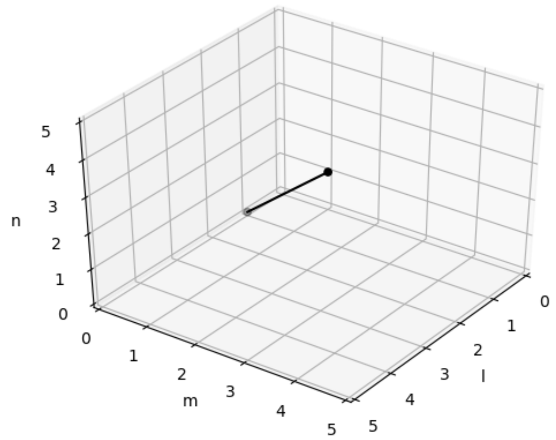
(Q3D-1) $(l+1)C_{l+1,m-1,0}-(m+1)C_{l-1,m+1,0}+C_{l,m,1}-C_{l,m-2,1}=0,\ l\geq 1, m\geq 2$



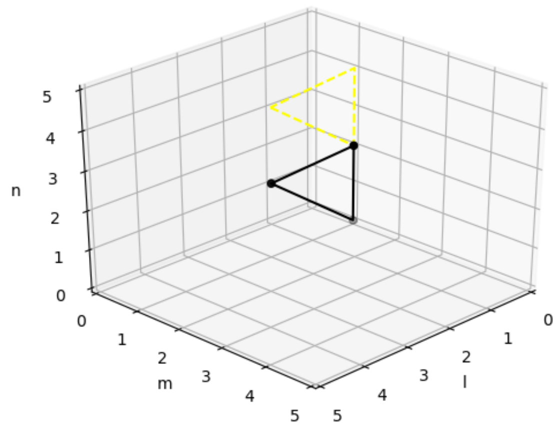
(Q3D-2) $(n+1)C_{0,0,n+1}=0,\ n\geq 0$



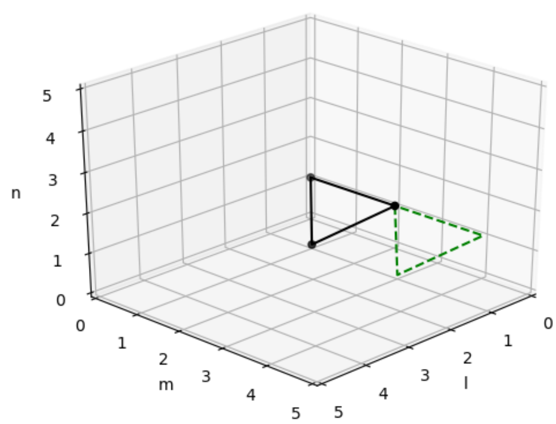
(Q3D-3) $C_{1,0,0} + C_{0,1,0} = 0$



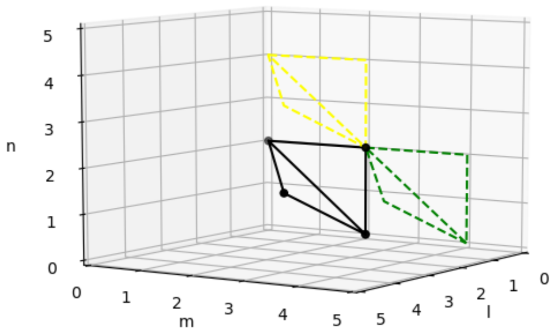
(Q3D-4) $C_{1,0,n} + C_{0,1,n-1} + (n+1)C_{0,1,n+1} = 0,\ n \geq 1$



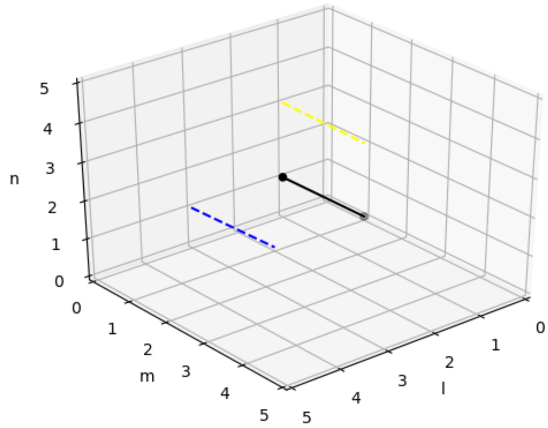
(Q3D-5) $C_{1,m-1,0} + C_{0,m,1} - C_{0,m-2,1} = 0,\ m \geq 2$



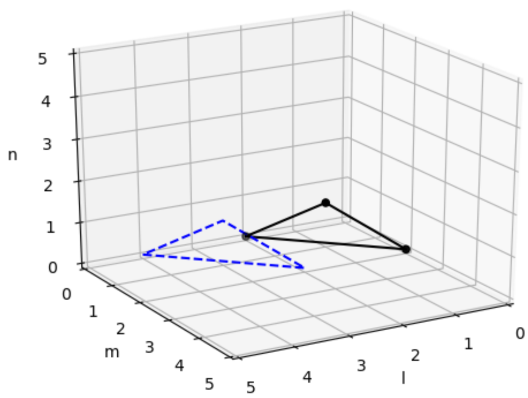
(Q3D-6) $C_{1,m-1,n} + mC_{0,m,n+1} + (n+1)C_{0,m,n+1} = 0,\ m \geq 2, n \geq 1$



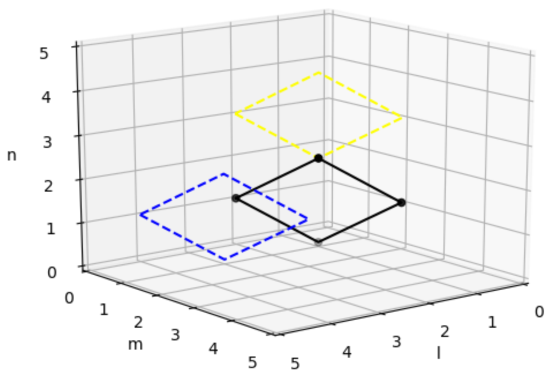
(Q3D-7) $-C_{l-1,1,n} + (n+1)C_{l,0,n+1} = 0,\ l \geq 1, n \geq 0$



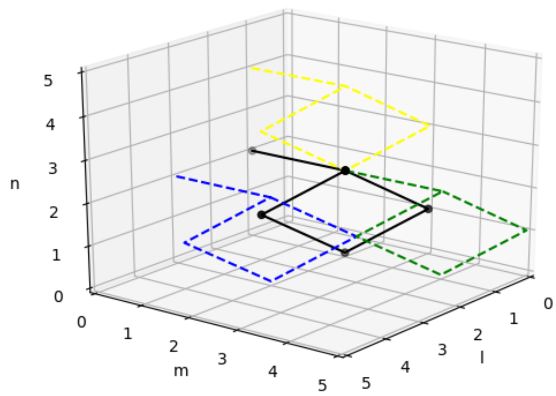
(Q3D-8) $(l+1)C_{l+1,0,0} - 2C_{l-1,2,0} + C_{l,1,1} = 0,\ l \geq 1$



(Q3D-9) $(l+1)C_{l+1,0,n} - 2C_{l-1,2,n} + C_{l,1,n-1} + (n+1)C_{l,1,n+1=0}$, $l \geq 1, n \geq 1$



(Q3D-10) $(l+1)C_{l+1,m-1,n} - (m+1)C_{l-1,m+1,n} + mC_{l,m,n-1}$
$+(n+1)C_{l,m,n+1} - (n+1)C_{l,m-2,n+1} = 0, l, n \geq 1, m \geq 2$



5.2 Equivalent matrix form

Interestingly, for this system, for some undetermined coefficients, there always exists an analytic solution around the origin. Let $M := l + m \geq 0$, where $f(x, y, z) = \sum_{l=0,m=0,n=0}^{\infty} C_{l,m,n}\, x^l y^m z^n$ for $n \geq 0$.

$M = 0$: $C_{0,0,0}$ free, $C_{0,0,n+1} = 0$ for $n \geq 0$.

$M = 1$: $C_{1,0,0}$, $C_{0,1,0}$ free

$$\begin{bmatrix} C_{1,0,1} \\ C_{0,1,1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} C_{1,0,0} \\ C_{0,1,0} \end{bmatrix}$$

$$\begin{bmatrix} C_{1,0,n+1} \\ C_{0,1,n+1} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{n+1} \\ -\frac{1}{n+1} & 0 \end{bmatrix} \begin{bmatrix} C_{1,0,n} \\ C_{0,1,n} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{n+1} \end{bmatrix} \begin{bmatrix} C_{1,0,n-1} \\ C_{0,1,n-1} \end{bmatrix}$$

Solving the above recurrence, we find the closed-form formula for $C_{1,0,N}$ and $C_{0,1,N}$, $N \geq 0$.

$$\implies \begin{cases} C_{1,0,2n} & = \frac{(-1)^n C_{1,0,0}}{2n(2n-2)\ldots 4\cdot 2},\ n \geq 0 \\ C_{1,0,2n+1} & = \frac{(-1)^n C_{0,1,0}}{(2n+1)(2n-1)\ldots 3\cdot 1},\ n \geq 0 \end{cases} \text{ and } \implies \begin{cases} C_{0,1,2n} & = \frac{(-1)^n C_{0,1,0}}{(2n-1)\ldots 3\cdot 1},\ n \geq 0 \\ C_{0,1,2n+1} & = \frac{(-1)^n C_{1,0,0}}{2n(2n-2)\ldots 4\cdot 2},\ n \geq 0 \end{cases}$$

$M = 2$: $C_{2,0,0}$, $C_{1,1,0}$ and $C_{0,2,0}$ free Let $\boldsymbol{v}_n^2 := \begin{bmatrix} C_{2,0,n} & C_{1,1,n} & C_{0,2,n} \end{bmatrix}^T$,

$$P_n^2 = \begin{bmatrix} 0 & \frac{1}{n} & 0 \\ -\frac{2}{n} & 0 & \frac{2}{n} \\ 0 & -\frac{1}{n} & 0 \end{bmatrix} \text{ and } Q_n^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{n} & 0 \\ 0 & 0 & -\frac{2}{n} \end{bmatrix}. \text{ Then}$$

$$\begin{aligned} \boldsymbol{v}_1^2 &= P_1^2 \boldsymbol{v}_0^2, \\ \boldsymbol{v}_{n+1}^2 &= P_{n+1}^2 \boldsymbol{v}_n^2 + Q_{n+1}^2 \boldsymbol{v}_{n-1}^2,\ n \geq 1 \end{aligned}$$

$\vdots$

For arbitrary $M \geq 3$: let $\boldsymbol{v}_n^M = \begin{bmatrix} C_{M,0,n} & C_{M-1,1,n} \ldots C_{1,M-1,n} & C_{0,M,n} \end{bmatrix}^T$, $n \geq 0$, and $\boldsymbol{u}_n^M = \begin{bmatrix} 0 & 0 & C_{M-2,0,n} & \ldots & C_{0,M-2,n} \end{bmatrix}^T$, $n \geq 1$.

$$P_n^M = \begin{bmatrix} 0 & \frac{1}{n} & & & & \\ -\frac{M}{n} & 0 & \frac{2}{n} & & & \\ 0 & -\frac{M-1}{n} & 0 & \frac{3}{n} & & \\ 0 & \ddots & \ddots & \ddots & & \\ & & -\frac{2}{n} & 0 & \frac{M}{n} \\ & & 0 & -\frac{1}{n} & 0 \end{bmatrix} \text{ and } Q_n^M = \begin{bmatrix} 0 & 0 & & \\ 0 & -\frac{1}{n} & & \\ 0 & 0 & -\frac{2}{n} & \\ & & \ddots & \\ 0 & \ldots & 0 & -\frac{M}{n} \end{bmatrix}.$$

$$\begin{aligned} \boldsymbol{v}_1^M &= P_1^M \boldsymbol{v}_0^M + \boldsymbol{u}_1^M \\ \boldsymbol{v}_{n+1}^M &= P_{n+1}^M \boldsymbol{v}_n^M + Q_{n+1}^M \boldsymbol{v}_{n-1}^M + \boldsymbol{u}_{n+1}^M,\ n \geq 1. \end{aligned}$$

Next, we show that there is no polynomial conserved quantities for system Q3D. First, we will need two lemmas.

Lemma 1. *For any $M \geq 3$ and $n \geq 1$, if $\boldsymbol{v}_n^M = \boldsymbol{v}_{n+1}^M = \boldsymbol{0}$ and $0 = \boldsymbol{u}_0^M = \boldsymbol{u}_1^M = \ldots = \boldsymbol{u}_n^M$, then $\boldsymbol{v}_0^M = \ldots = \boldsymbol{v}_{n-1}^M = \boldsymbol{0}$.*

Proof. We will prove by induction on n and fix M.
Base case: $n = 1$, if $\boldsymbol{v}_1^M = \boldsymbol{v}_2^M = \boldsymbol{0}$, then

$$\boldsymbol{v}_2^M = P_2^M \boldsymbol{v}_1^M + Q_2^M \boldsymbol{v}_0^M + \boldsymbol{u}_2^M \implies \boldsymbol{0} = \begin{bmatrix} 0 & 0 & \\ 0 & -\frac{1}{2} & \\ \ddots & \ddots & \\ 0 & \ldots & -\frac{M}{2} \end{bmatrix} \boldsymbol{v}_0^M \implies \boldsymbol{v}_0^M = \begin{bmatrix} C \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ for some } C \in \mathbb{R}$$

and

$$\boldsymbol{v}_1^M = P_1^M \boldsymbol{v}_0^M + \boldsymbol{u}_1^M \implies \boldsymbol{0} = \begin{bmatrix} 0 & 0 & & \\ -M & 0 & 2 & \\ \ddots & \ddots & \ddots & \\ & -2 & 0 & M \\ & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} C \\ 0 \\ \vdots \\ 0 \end{bmatrix} \implies C = 0.$$

So $\boldsymbol{v}_0^M = \mathbf{0}$.

Hypothesis: assume that for $n = k \geq 2$, $\boldsymbol{v}_k^M = \boldsymbol{v}_{k+1}^M = \mathbf{0} \implies \boldsymbol{v}_0^M = \ldots = \boldsymbol{v}_{k-1}^M = \mathbf{0}$.
Induction step: $n = k+1$, suppose that $\boldsymbol{v}_{k+1}^M = \boldsymbol{v}_{k+2}^M = \mathbf{0}$.

(1)

$$\boldsymbol{v}_{k+2}^M = P_{k+2}^M \boldsymbol{v}_{k+1}^M + Q_{k+2}^M \boldsymbol{v}_k^M + \boldsymbol{u}_{k+2}^M$$

$$\iff \mathbf{0} = \begin{bmatrix} 0 & 0 & & \\ 0 & -\frac{1}{k+2} & & \\ & & \ddots & \\ 0 & \ldots & 0 & -\frac{M}{k+2} \end{bmatrix} \boldsymbol{v}_k^M \implies \boldsymbol{v}_k^M = \begin{bmatrix} C \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ for some } C \in \mathbb{R}.$$

(2)

$$\boldsymbol{v}_{k+1}^M = P_{k+1}^M \boldsymbol{v}_k^M + Q_{k+1}^M \boldsymbol{v}_{k-1}^M + \boldsymbol{u}_{k+1}^M$$

$$\iff \mathbf{0} = \begin{bmatrix} 0 & \frac{1}{k+1} & & & \\ -\frac{M}{k+1} & 0 & \frac{2}{k+1} & & \\ \ddots & \ddots & \ddots & & \\ & -\frac{2}{k+1} & 0 & \frac{M}{k+1} \\ & 0 & -\frac{1}{k+1} & 0 \end{bmatrix} \begin{bmatrix} C \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & & \\ 0 & -\frac{1}{k+1} & & \\ & & \ddots & \\ 0 & \ldots & 0 & -\frac{M}{k+1} \end{bmatrix} \boldsymbol{v}_{k-1}^M$$

$$\iff \boldsymbol{v}_{k-1}^M = \begin{bmatrix} D \\ -MC \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ for some } D \in \mathbb{R}.$$

(3)

$$\boldsymbol{v}_k^M = P_k^M \boldsymbol{v}_{k-1}^M + Q_k^M \boldsymbol{v}_{k-2}^M + \boldsymbol{u}_k^M.$$

$$\iff \begin{bmatrix} C \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{k} & & \\ -\frac{M}{k} & 0 & \frac{2}{k} & \\ \ddots & \ddots & \ddots & \\ & -\frac{2}{k} & 0 & \frac{M}{k} \\ & 0 & -\frac{1}{k} & 0 \end{bmatrix} \begin{bmatrix} D \\ -MC \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & & \\ 0 & -\frac{1}{k} & & \\ & & \ddots & \\ & & & -\frac{M}{k} \end{bmatrix} \boldsymbol{v}_{k-2}^M.$$

$$\implies C = -\frac{M}{k} C \implies C = 0 \implies \boldsymbol{v}_k^M.$$

Now, $\boldsymbol{v}_k^M = \boldsymbol{v}_{k+1}^M = \mathbf{0}$, by the induction hypothesis, we have $\boldsymbol{v}_0^M = \ldots = \boldsymbol{v}_{k-1}^M = \mathbf{0}$. Hence, $\boldsymbol{v}_0^M = \ldots = \boldsymbol{v}_k^M = \mathbf{0}$. □

Lemma 2. *For any $M = 1$ or 2 and all $n \neq 1$, if $\boldsymbol{v}_n^M = \boldsymbol{v}_{n+1}^M = \mathbf{0}$, then $\boldsymbol{v}_0^M = \boldsymbol{v}_{n-1}^M = \mathbf{0}$.*

Proof. Since when $M = 1$ or 2, we have $\boldsymbol{v}_1^M = P_1^M \boldsymbol{v}_0^M$ and $\boldsymbol{v}_{n+1}^M = P_{n+1}^M \boldsymbol{v}_n^M + Q_{n+1}^M \boldsymbol{v}_{n-1}^M$, the proof is similar to Lemma 1. □

Now, we are ready to prove our main result.

Theorem 5. *For system (Q3D), assume a time-independent conserved quantity is of the form $f(x, y, z) = \psi(x, y, z, t)$, i.e., $\frac{d\psi}{dt} = 0$. Then it is either the case $f(x, y, z) = C$ for some constant $C \in \mathbb{R}$ or that there exists some $M \geq 1$ such that for any $n \geq 0$, $\boldsymbol{v}_n^M \neq 0$ or $\boldsymbol{v}_{n+1}^M \neq 0$, i.e., analytic conserved quantity.*

Proof. If $C_{0,0,0} = C$ and $C_{l,m,n} = 0$ otherwise, it is the first case of the statement. If not, then there is some $N_0 \geq 1$ such that $\boldsymbol{v}^0_{N_0} \neq 0$ or some $n \geq 1$ such that $\boldsymbol{v}^M_{N_M} \neq \mathbf{0}$ for some $M \geq 1$.

From the previous lattice equation (Q3D-2), we know that $\boldsymbol{v}^0_n = \mathbf{0}$ for all $n \geq 1$. So for some $L \geq 1$ and some $N_L \geq 1$, $\boldsymbol{v}^L_{N_L} \neq 0$. Define the nonempty set $S := \{L \geq 1 : \exists N_L \geq 1, \boldsymbol{v}^L_{N_L} \neq \mathbf{0}\}$. We now prove case two by contracdiction.

Let $M \geq 1$ be the smallest integer in the set S, so $\boldsymbol{v}^M_{N_M} \neq \mathbf{0}$ for some $N_M \geq 1$, and $\forall 1 \leq k < M$, $\boldsymbol{v}^M_n = \mathbf{0}$ for all $n \geq 0$ if $M > 1$. Suppose that $\boldsymbol{v}^M_n = \boldsymbol{v}^M_{n+1} = \mathbf{0}$ for some $n \geq 0$. By lemma 1 and 2, $\boldsymbol{v}^M_p = \mathbf{0}$ for all $p \geq 0$. So either way, there is a contradiction with the choice of S. Thus, for such M, for any $n \geq 0$, it must have $\boldsymbol{v}^M_n \neq \mathbf{0}$ or $\boldsymbol{v}^M_{n+1} \neq \mathbf{0}$. □

Unfortunately, we were unable to a find closed-form formula to the analytic coefficients, but we are able to derive some partial results.

Proposition 1. *A conserved quantity for the system Q3D is of the from* $\psi(x, y, z) = \sum_{M=0}^{\infty} f_M(x, y, z)$, *where* $f_M = x^l y^{M-l} \sum_{n=0}^{\infty} C_{l,M-l,n}\, z^n$ *and* $C_{l,M-l,0}$ *is a free parameter for all* $M \geq 0$ *and all* $l \leq M$. *Moreover,* f_1 *has the closed-form formula*

$$\begin{aligned} f_1 = x \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2^n n!} C_{1,0,0} + \frac{(-1)^n (2n+2)!}{2^{n+1}(n+1)!} C_{0,1,0}\, z^{2n+1} \right] \\ + y \sum_{n=0}^{\infty} \left[\frac{(-1)^n (2n)!}{2^n n!} C_{0,1,0}\, z^{2n} + \frac{(-1)^{n+1}}{2^n n!} C_{1,0,0}\, z^{2n+1} \right] \end{aligned}$$

Chapter 6

Conclusion

In this research project, we introduced a systematic approach to derive conserved quantities for various dynamical systems of the form $\psi(\boldsymbol{x}) = f(\boldsymbol{x})e^{-kt}$, where $f(\boldsymbol{x})$ is polynomial or analytic. In particular, we needed to find the coefficients of $f(\boldsymbol{x})$. By computation, we found relations among the coefficients in the form of recurrence equations. To study them more systematically, we drew the diagrams on lattices, and rewrote the equations in their equivalent matrix forms. This reduced to solving the matrix equations, which is essentially a linear algebra problem. To illustrate our approach, we applied it to three systems: the general quadratic system (Q2D), the Lorenz system (L3D) and a nondissipative chaotic system (Q3D).

In our study of Q2D system, we were able to find conserved quantity of some special form, and we have an algorithm and illustrated one example for finding conserved quantities. For the Lorenz system, it turned out that generally no conserved quantity exists unless there are some constraints on the parameters. Under those constraints, we applied our method to recover the six known conserved quantities of polynomial form that appear previously in literature. Lastly, we also studied a nondissipative chaotic system, and revealed that it has no polynomial conserved quantity.

For future study, we are interested in applying the "lattice approach" to higher-order polynomial 2D systems and more chaotic systems. Meanwhile, we will also be interested in investigating other approaches to find conserved quantities for dynamical systems.

Appendix

A Additional proof of lemmas

Theorem 6 (Criterion for Hamiltonian systems in 2D). *Recall that a Hamiltonian system in* $\mathbb{R}^2$ *has the form*

$$\begin{aligned} \dot{x} &= -\frac{\partial \mathcal{H}}{\partial y} \\ \dot{y} &= \frac{\partial \mathcal{H}}{\partial x} \end{aligned}, \tag{1}$$

where $\mathcal{H}(x,\, y)$ *is the Hamiltonian equation. Suppose the system* (Q2D) *is a Hamiltonian system in* $\mathbb{R}^2$*, then the condition* $A = -G$*,* $I = -2C$ *and* $J = -2D$ *must be satisfied. Moreover, the Hamiltonian is given by*

$$\mathcal{H}(x,\, y) = \frac{F}{2}x^2 - Axy - \frac{B}{2}y^2 + \frac{H}{3}x^3 - Cx^2y + Jxy^2 - \frac{E}{3}y^3 \tag{2}$$

Proof. Substituting (1) into (Q2D) yields

$$-\frac{\partial \mathcal{H}}{\partial y} = Ax + By + Cx^2 + Dxy + Ey^2 \tag{3}$$

$$\frac{\partial \mathcal{H}}{\partial x} = Fx + Gy + Hx^2 + Ixy + Jy^2. \tag{4}$$

Integrating (3) with respect to y gives

$$\mathcal{H} = -Axy - \frac{B}{2}y^2 - Cx^2y - \frac{D}{2}xy^2 - \frac{E}{3}y^3 + C(x). \tag{5}$$

Differentiaing (5) implies

$$Fx + Gy + Hx^2 + Ixy + Jy^2 = \frac{\partial \mathcal{H}}{\partial x} = -Ay - 2Cxy - \frac{D}{2}y^2 + C'(x).$$

Hence, we can derive $A = -G$, $I = -2C$ and $D = -2J$. Furthermore, the conserved quantity is given by

$$\mathcal{H}(x,\, y) = \frac{F}{2}x^2 - Axy - \frac{B}{2}y^2 + \frac{H}{3}x^3 - Cx^2y + Jxy^2 - \frac{E}{3}y^3.$$

□

B Derivation of six conserved quantities for the Lorenz system

Poly of order 4

July 12, 2020 8:38 PM

Assume $k \neq 0,\ s > 0,\ b \geq 0,\ r \geq 0$.

$C_{6,0,n} = \cdots = C_{0,6,n} = 0,\ n \geq 0 \Rightarrow C_{4,0,n} = C_{3,1,n} = C_{2,2,n} = C_{1,3,n} = C_{0,4,n} = 0,\ n \geq 1$

$\Rightarrow C_{2,0,n} = C_{1,1,n} = C_{0,2,n} = 0,\ n \geq 3.$

$$\begin{cases} C_{1,1,2} = 0 \\ 2C_{0,2,2} - 4C_{0,0,4} = 0 \\ C_{3,1,0} = 0 \\ 2C_{2,2,0} - 2C_{2,0,2} = 0 \\ 3C_{1,3,0} - 2C_{1,1,2} = 0 \\ 4C_{0,4,0} - 2C_{0,2,2} = 0 \end{cases} \Rightarrow \begin{cases} C_{1,1,2} = 0 \\ C_{3,1,0} = C_{1,3,0} = 0 \\ C_{0,2,2} = 2C_{0,4,0} = 2C_{0,0,4} \\ C_{2,2,0} = C_{2,0,2}. \end{cases}$$

$m=4,\ n=0$:

$$\begin{bmatrix} -(4s+k) & 0 & 0 \\ 4s & 2r & 0 \\ 0 & -(2s+2+k) & 0 \\ 0 & 2s & 4r \\ 0 & 0 & -(4+k) \end{bmatrix} \begin{bmatrix} C_{4,0,0} \\ C_{2,2,0} \\ C_{0,4,0} \end{bmatrix} = \begin{bmatrix} 0 \\ -C_{2,0,1} \\ -C_{1,1,1} \\ -C_{0,2,1} \\ 0 \end{bmatrix}$$

($C_{2,2,0} = C_{2,0,2}$, $C_{0,4,0} = C_{0,0,4}$)

$m=2,\ n=2$:

$$\begin{bmatrix} -(2s+k+2b) & 0 \\ 2s & 2r \\ 0 & -(2+k+2b) \end{bmatrix} \begin{bmatrix} C_{2,0,2} \\ C_{0,2,2} \end{bmatrix} = \begin{bmatrix} C_{1,1,1} \\ 2C_{0,2,1} - 3C_{0,0,3} \\ 0 \end{bmatrix}$$

($C_{0,2,2} = \frac{1}{2} C_{0,0,4}$)

$$\Rightarrow \begin{cases} (k+4b)C_{0,0,4} = 0 \\ (2+k+2b)\frac{1}{2}C_{0,0,4} = 0 \\ (4+k)C_{0,0,4} = 0. \end{cases} \Rightarrow \begin{cases} k=-4 \\ b=1 \end{cases} \text{ or } C_{0,0,4} = 0$$

$$\begin{cases} (2s+2+k)C_{2,0,2} = C_{1,1,1} \\ (2s+k+2b)C_{2,0,2} = -C_{1,1,1} \end{cases} \Rightarrow \begin{array}{l} 2s+k+1+b = 0 \\ C_{1,1,1} = (2s+2+k)C_{2,0,2} \end{array} \text{ or } C_{2,0,2} = C_{1,1,1} = 0.$$

$$(4s+k)C_{4,0,0} = 0 \Rightarrow 4s+k = 0 \text{ or } C_{4,0,0} = 0.$$

$$\begin{cases} 4s\, C_{4,0,0} + 2r\, C_{2,0,2} = -C_{2,0,1} \\ 2s\, C_{2,0,2} + 4r\, C_{0,0,4} = -C_{0,2,1} \\ 2s\, C_{2,0,2} + 2r \frac{1}{2} C_{0,0,4} = 2C_{0,2,1} - 3C_{0,0,3} \end{cases} \Rightarrow \begin{cases} C_{0,2,1} = C_{0,0,3} - r\, C_{0,0,4} \\ C_{2,0,2} = -\frac{1}{2s} C_{0,0,3} - \frac{3r}{2s} C_{0,0,4}. \\ C_{4,0,0} = -\frac{1}{4s} C_{2,0,1} + \frac{r}{4s^2} C_{0,0,3} + \frac{3r^2}{4s^2} C_{0,0,4} \end{cases}$$

$m=2,\ n=1$:

$$\begin{bmatrix} -(2s+k+b) & r & 0 \\ 2s & -(s+1+k+b) & 2r \\ 0 & s & -(2+k+b) \end{bmatrix} \begin{bmatrix} C_{2,0,1} \\ C_{1,1,1} \\ C_{0,2,1} \end{bmatrix} = \begin{bmatrix} C_{1,1,0} \\ 2C_{0,2,0} - 2C_{0,0,2} \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} C_{1,1,1} = \frac{2+k+b}{s} C_{0,2,1} \\ C_{2,0,1} = \frac{s+1+k+b}{2s} C_{1,1,1} - \frac{r}{s} C_{0,2,1} + \frac{1}{s} C_{0,2,0} - \frac{1}{s} C_{0,0,2} \\ C_{1,1,0} = -(2s+k+b)C_{2,0,1} + r\, C_{1,1,1} \end{cases}$$

$\rightarrow C_{1,1,1} = \frac{2+k+b}{s}\left(C_{0,0,3} - r\, C_{0,2,4}\right)$

$C_{2,0,\ldots} = \frac{(2+k+b)(s+1+k+b) - 2sr}{\ldots} \ldots$

$$\frac{\quad}{s}\left(C_{0,0,3} - r\, C_{0,0,4}\right)$$

$$C_{2,0,1} = \frac{(2+k+b)(s+1+k+b) - 2sr}{2s^2}\left(C_{0,0,3} - r\,C_{0,0,4}\right) + \frac{1}{s}\left(C_{0,2,0} - C_{0,0,2}\right)$$

$$C_{1,1,0} = \frac{(s+1+k+b)\left[4rs - (2+k+b)(2s+k+b)\right]}{2s^2}\left(C_{0,0,3} - r\,C_{0,0,4}\right) - \frac{2s+k+b}{s}\left(C_{0,2,0} - C_{0,0,2}\right).$$

$m=2,\ n=0$:

$$\begin{bmatrix} -(2s+k) & r & 0 \\ 2s & -(s+1+k) & 2r \\ 0 & s & -(2+k) \end{bmatrix}\begin{bmatrix} C_{2,0,0} \\ C_{1,1,0} \\ C_{0,2,0} \end{bmatrix} = \begin{bmatrix} 0 \\ -C_{0,0,1} \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -(2s+k)\,C_{2,0,0} + r\,C_{1,1,0} = 0. \\ C_{2,0,0} = \dfrac{s+1+k}{2s}\,C_{1,1,0} - \dfrac{r}{s}\,C_{0,2,0} - \dfrac{1}{2s}\,C_{0,0,1} \\ C_{1,1,0} = \dfrac{2+k}{s}\,C_{0,2,0}. \end{cases}$$

$$\Rightarrow \quad C_{1,1,0} = \frac{2+k}{s}\,C_{0,2,0}$$

$$C_{2,0,0} = \frac{(2+k)(s+1+k) - 2sr}{2s^2}\,C_{0,2,0} - \frac{1}{2s}\,C_{0,0,1}$$

$$(2s+k)\,C_{2,0,0} = \frac{r(2+k)}{s}\,C_{0,2,0}.$$

We have the following equations:

$$C_{1,1,2} = 0$$

$$C_{3,1,0} = C_{1,3,0} = 0$$

$$C_{0,2,2} = 2\,C_{0,4,0} = 2\,C_{0,0,4}$$

$$C_{2,2,0} = C_{2,0,2}.$$

$$\begin{cases} k=-4 \\ b=1 \end{cases} \quad \text{or} \quad C_{0,0,4} = 0$$

$2s+k+1+b=0,\ C_{1,1,1} = (2s+2+k)\,C_{2,0,2}$ or $C_{2,0,2} = C_{1,1,1} = 0.$

$4s+k=0$ or $C_{4,0,0} = 0.$

$$C_{0,2,1} = C_{0,0,3} - r\,C_{0,0,4}$$

$$C_{2,0,2} = -\frac{1}{2s}\,C_{0,0,3} - \frac{3r}{2s}\,C_{0,0,4}.$$

$$C_{4,0,0} = -\frac{1}{4s}\,C_{2,0,1} + \frac{r}{4s^2}\,C_{0,0,3} + \frac{3r^2}{4s^2}\,C_{0,0,4}$$

$$C_{1,1,1} = \frac{2+k+b}{s}\,C_{0,0,3} - \frac{r(2+k+b)}{s}\,C_{0,0,4}$$

$$C_{2,0,1} = \frac{(2+k+b)(s+1+k+b) - 2sr}{2s^2}\left(C_{0,0,3} - r\,C_{0,0,4}\right) + \frac{1}{s}\left(C_{0,2,0} - C_{0,0,2}\right)$$

$$C_{1,1,0} = \frac{(s+1+k+b)\left[4rs - (2+k+b)(2s+k+b)\right]}{2s^2}\left(C_{0,0,3} - r\,C_{0,0,4}\right) - \frac{2s+k+b}{s}\left(C_{0,2,0} - C_{0,0,2}\right)$$

$$C_{1,1,0} = \frac{2+k}{s}\,C_{0,2,0}$$

$$C_{2,0,0} = \frac{(2+k)(s+1+k) - 2sr}{2s^2}\,C_{0,2,0} - \frac{1}{2s}\,C_{0,0,1}$$

$(2s+k)\, C_{2,0,0} = \frac{r(2+k)}{s} C_{0,2,0}.$

Suppose $k+b=0 \iff k=-b$:

$C_{0,0,1} = C$ free

$C_{0,0,4}=0$

$C_{1,1,1}=C_{2,0,2}=0.$

$b=4s$ or $C_{4,0,0}=0.$

$C_{0,2,1}=0$

$C_{4,0,0} = -\frac{1}{4s} C_{2,0,1}$

$C_{2,0,1} = \frac{1}{s} C_{0,2,0}.$

$C_{1,1,0} = -2\, C_{0,2,0}$

$C_{1,1,0} = \frac{2-b}{s} C_{0,2,0}$

$C_{2,0,0} = \frac{(2-b)(s+1-b)-2sr}{2s^2} C_{0,2,0} - \frac{1}{2s} C$

$(2s-b)\, C_{2,0,0} = \frac{r(2-b)}{s} C_{0,2,0}.$

if $b=2s$: $C_{0,0,1}=C$

$0 = C_{0,0,4} = C_{1,1,1} = C_{2,0,2} = C_{4,0,0} = C_{0,2,1}$
$= C_{2,0,1} = C_{0,2,0} = C_{1,1,0}$

$C_{2,0,0} = -\frac{1}{2s} C.$

$\therefore C_{2,0,0} = -\frac{1}{2s} C_{0,0,1}. \Rightarrow f(x,y,z) = x^2 - 2sz.$

if $b=4s$: $C_{0,0,1}=C$

$0 = C_{0,0,4} = C_{1,1,1} = C_{2,0,2} = C_{0,2,1} = C_{2,0,2}$

$C_{4,0,0} = D,\ C_{2,0,1} = -4sD,\ C_{0,2,0} = -4s^2D,$

$C_{1,1,0} = 8s^2D = (-4s)(2-4s)D \iff \boxed{s=1,\ b=4}.$

$C_{2,0,0} = 4rD - 8D - \frac{1}{2}C = -4rD. \iff C = 16(r-1)D$

$\therefore C_{0,0,1} = 16(r-1)D,\ C_{0,2,0} = -4D,\ C_{1,1,0} = 8D,$
$C_{2,0,0} = -4rD,\ C_{2,0,1} = -4D,\ C_{4,0,0} = D.$

$\Rightarrow f(x,y,z) = x^4 - 4x^2z - 4rx^2 + 8xy - 4y^2 + 16(r-1)z.$

else $b \neq 2s, 4s$: $C_{0,0,1} = C$

$0 = C_{0,0,4} = C_{1,1,1} = C_{2,0,2} = C_{4,0,0} = C_{0,2,1}$
$= C_{2,0,1} = C_{0,2,0} = C_{1,1,0} = C_{2,0,0}$

$C=0.$

$\therefore f(x,y,z)=0.$ No sol.

Suppose $k+2b=0 \iff k=-2b.$

$C_{0,0,2} = C$ free

$C_{0,0,4}=0$

$\begin{cases} 2s+1-b=0 \\ C_{1,1,1}=(2s+2-2b)\, C_{2,0,2} \end{cases}$ or $C_{2,0,2}=C_{1,1,1}=0.$

$b=2s$ or $C_{4,0,0}=0$

$C_{0,2,1}=0$

$C_{2,0,2}=0$

$C_{4,0,0} = -\frac{1}{4s} C_{2,0,1}$

$C_{1,1,1}=0$

$C_{2,0,1} = \frac{1}{s}(C_{0,2,0} - C)$

$C_{1,1,0} = -\frac{2s-b}{s}(C_{0,2,0} - C)$

$C_{1,1,0} = \frac{2-2b}{s} C_{0,2,0}$

$C_{2,0,0} = \frac{(2-2b)(s+1-2b)-2sr}{2s^2} C_{0,2,0}$

$(2s-2b)\, C_{2,0,0} = \frac{r(2-2b)}{s} C_{0,2,0}$

if $b=2s$: $C_{0,0,2}=C$

$0 = C_{0,0,4} = C_{2,0,2} = C_{1,1,1} = C_{0,2,1}$

$C_{4,0,0}=D,\ C_{2,0,1} = -4sD,\ C_{0,2,0} = C - 4s^2D$

$C_{1,1,0} = 0 = \frac{2-4s}{s}C - 8s(1-2s)D$

$\iff C = 4s^2D$

$\begin{cases} C_{2,0,0} = \frac{(1-2s)(1-3s)-sr}{s^2}(C-4s^2D) \\ C_{2,0,0} = \frac{r(2s-1)}{s^2}(C-4s^2D). \end{cases} \Rightarrow C_{2,0,0}=0.$

$\therefore C_{4,0,0}=D,\ C_{2,0,1} = -4sD,\ C_{0,0,2} = 4s^2D.$

$\Rightarrow f(x,y,z) = (x^2-2sz)^2$

else $b \neq 2s$: $C_{0,0,2} = C$

$0 = C_{0,0,4} = C_{4,0,0} = C_{0,2,1} = C_{2,0,2} = C_{2,0,1} = C_{1,1,1}$

$C_{0,2,0} = C$

$C_{1,1,0} = 0 = \frac{2-2b}{s} C \iff \boxed{b=1}$

$\begin{cases} C_{2,0,0} = -\frac{r}{s} C \\ (s-1)\, C_{2,0,0} = 0 \end{cases} \Rightarrow$

if $r=0$: $C_{0,0,2}=C,\ C_{0,2,0}=C$

$(s-1)C_{2,0,0}=0 \Rightarrow$

if $r=0$: $C_{0,0,2}=C$, $C_{0,2,0}=C$

$\therefore f(x,y,z)=y^2+z^2$.

else $r\neq 0$: $C_{0,0,2}=C=C_{0,2,0}$, $C_{2,0,0}=-\frac{r}{s}C\neq 0$

$\boxed{s=1}$

$\therefore f(x,y,z)=-rx^2+y^2+z^2$

Suppose $k+3b=0 \iff k=-3b$.

$C_{0,0,3}=C$ free

$C_{0,0,4}=0$

$\begin{cases} 2s+1-2b=0 \\ C_{1,1,1}=(2s+2-3b)C_{2,0,2} \end{cases}$ or $C_{2,0,2}=C_{1,1,1}=0$

$4s-3b=0$ or $C_{4,0,0}=0$

$C_{0,2,1}=C$

$C_{2,0,2}=-\frac{1}{2s}C$

$C_{4,0,0}=-\frac{1}{4s}C_{2,0,1}+\frac{r}{4s^2}C$

$C_{1,1,1}=\frac{2-2b}{s}C$

$C_{2,0,1}=\frac{(2-2b)(s+1-2b)-2sr}{2s^2}C+\frac{1}{s}C_{0,2,0}$

$C_{1,1,0}=\frac{(s+1-2b)[2rs-(2-2b)(s-b)]}{s^2}C-\frac{2s-2b}{s}C_{0,2,0}$

$C_{1,1,0}=\frac{2-3b}{s}C_{0,2,0}$

$C_{2,0,0}=\frac{(2-3b)(s+1-3b)-2sr}{2s^2}C_{0,2,0}$

$(2s-3b)C_{2,0,0}=\frac{r(2-3b)}{s}C_{0,2,0}$.

if $b=s+\frac{1}{2}$:

if $s=\frac{3}{2}, b=2$: $C_{0,0,3}=C=C_{0,2,1}$,

$0=C_{0,0,4}$

$\begin{cases} C_{1,1,1}=-C_{2,0,2} \\ C_{2,0,2}=-\frac{1}{3}C,\ C_{1,1,1}=-\frac{4}{3}C \end{cases} \Rightarrow$ $C=0$, $C_{1,1,1}=C_{2,0,2}=0$

$0=C_{0,2,1}=C_{0,0,3}$

$C_{4,0,0}=-\frac{1}{6}C_{2,0,1}$

$C_{2,0,1}=\frac{2}{3}C_{0,2,0}=C_{1,1,0}=-\frac{8}{3}C_{0,2,0}$

$\Rightarrow 0=C_{0,2,0}=C_{1,1,0}=C_{2,0,1}=C_{4,0,0}=C_{2,0,0}$

$\therefore$ No sol.

else: $C_{0,0,3}=C=C_{0,2,1}$, $C_{2,0,1}=\frac{r}{s}C$

$0=C_{0,0,4}=C_{4,0,0}$

$C_{1,1,1}=(\frac{1}{2}-s)C_{2,0,2}$

$C_{1,1,1}=\frac{1-2s}{s}C$

$C_{2,0,2}=-\frac{1}{2s}C$

} contradiction $\therefore$ No sol.

else $b\neq s+\frac{1}{2}$: $0=C_{2,0,2}=C_{1,1,1}$, $C_{2,0,2}=-\frac{1}{2s}C$ } $\Rightarrow C=0$

$0=C_{0,0,3}=C_{0,0,4}=C_{0,2,1}$

$b=\frac{4}{3}s$ or $C_{4,0,0}=0$

$C_{4,0,0}=-\frac{1}{4s}C_{2,0,1}=-\frac{1}{4s^2}C_{0,2,0}$

$C_{2,0,1}=\frac{1}{s}C_{0,2,0}$

$C_{1,1,0}=\frac{2b-2s}{s}C_{0,2,0}$

$C_{1,1,0}=\frac{2-3b}{s}C_{0,2,0}$

} $b=\frac{2s+2}{5}$ or $C_{1,1,0}=C_{0,2,0}=0$

if $b\neq\frac{4}{3}s$ or $b\neq\frac{2s+2}{5}$

$0=C_{2,0,1}=C_{4,0,0}$ $\therefore$ No sol.

else $b=\frac{4}{3}s=\frac{2s+2}{5}$:

$s=\frac{3}{7}$, $b=\frac{4}{7}$

$\begin{cases} C_{2,0,0}=\frac{(2-4s)(1-3s)-2sr}{2s^2}C_{0,2,0} \\ s(-2s)C_{2,0,0}=r(2-4s)C_{0,2,0} \end{cases}$

$\Rightarrow r=2s-1=-\frac{1}{7}<0$ $\therefore$ No sol.

Suppose $k+4b=0 \iff k=-4b$.

if $k=-4, b=1$: $C_{0,0,4}=C$ free

$C_{0,0,4}=C$, let $C_{0,2,0}=D$

$0=C_{1,1,\ldots}=C_{\ldots}=C$

Suppose $k+4b=0 \iff k=-4b$.

if $k=-4, b=1$: $C_{0,0,4}=C$ free

$\begin{cases} 2s-2=0 \\ C_{1,1,1}=(2s-2)C_{2,0,2}=0 \end{cases}$ or $C_{2,0,2}=C_{1,1,1}=0$.

$4s-4=0$ or $C_{4,0,0}=0$

$C_{0,2,1}=-rC$

$C_{2,0,2}=-\frac{3r}{2s}C$

$C_{4,0,0}=-\frac{1}{4s}C_{2,0,1}+\frac{3r^2}{4s^2}C$

$C_{1,1,1}=\frac{r}{s}C=0 \iff r=0$ or $C=0$

$C_{2,0,1}=\frac{1}{s}C_{0,2,0}$

$C_{1,1,0}=-\frac{2s-3}{s}C_{0,2,0}$

$C_{1,1,0}=-\frac{2}{s}C_{0,2,0}$

$C_{2,0,0}=\frac{-2(s-3)-2sr}{2s^2}C_{0,2,0}$

$(2s-4)C_{2,0,0}=-\frac{2r}{s}C_{0,2,0}$

$C_{0,0,4}=C$, let $C_{0,2,0}=D$

$0=C_{1,1,1}=C_{0,2,1}=C_{2,0,2}$

$C_{2,0,1}=\frac{1}{s}D$

$C_{4,0,0}=-\frac{1}{4s^2}D$

$r=0$ or $C=0$

$\Rightarrow$ $s=\frac{5}{2}$ or $C_{1,1,0}=C_{0,2,0}=D=0$.

if $r=0$:

$\begin{cases} C_{2,0,0}=\frac{3-s}{s^2}D \\ (2s-4)C_{2,0,0}=0 \\ s=\frac{5}{2} \text{ or } D=0 \end{cases} \Rightarrow \begin{cases} C_{2,0,0}=0 \\ D=0 \end{cases}$.

$\therefore 0=C_{2,0,0}=C_{0,2,0}=C_{1,1,0}=C_{4,0,0}$

$=C_{2,0,1}=C_{2,0,2}=C_{0,2,1}=C_{1,1,1}$

$C_{0,2,2}=2C_{0,4,0}=2C_{0,0,4}=2C$

$\Rightarrow f(x,y,z)=(y^2+z^2)^2$.

else $C=0$:

$0=C_{0,0,4}=C_{0,4,0}=C_{0,2,2}$

$\begin{cases} s=\frac{5}{2} \text{ or } D=0 \\ C_{2,0,0}=\frac{3-s-sr}{s^2}D \\ (2s-4)C_{2,0,0}=-\frac{2r}{s}D \end{cases} \Rightarrow$ ~~$s=\frac{5}{2}, r=\frac{1}{2}$~~ $D=0=C_{2,0,0}$

$0=C_{2,0,0}=C_{0,2,0}=C_{1,1,0}=C_{4,0,0}$

$=C_{2,0,1}=C_{2,0,2}=C_{0,2,1}=C_{1,1,1}$

$\therefore$ No sol.

else:

$C_{0,0,4}=0$

$4s-4b=0$ or $C_{4,0,0}=0$

$C_{0,2,1}=0$

$C_{2,0,2}=0=C_{1,1,1}$

$C_{4,0,0}=-\frac{1}{4s}C_{2,0,1}$

$C_{1,1,1}=0$

$C_{2,0,1}=\frac{1}{s}C_{0,2,0}$

$C_{1,1,0}=-\frac{2s-3b}{s}C_{0,2,0}$

$C_{1,1,0}=\frac{2-4b}{s}C_{0,2,0}$

$\Big\}$ $b=\frac{2+2s}{7}$ or $C_{1,1,0}=C_{0,2,0}=0$

$C_{2,0,0}=\frac{(2-4b)(s+1-4b)-2sr}{2s^2}C_{0,2,0}$

$(2s-4b)C_{2,0,0}=\frac{r(2-4b)}{s}C_{0,2,0}$.

if $b=\frac{2+2s}{7}$: $r=-\frac{(3s-4)(4s-3)}{49s}$

Suppose $C_{0,2,0}\neq 0 \Rightarrow C_{4,0,0}\neq 0$

$\therefore 4s-4b=0 \Rightarrow s=b$.

$\therefore s=\frac{2}{5}, b=\frac{2}{5}, r=-\frac{1}{5}<0$

$\therefore$ No sol.

else: $0=C_{0,0,4}=C_{0,2,0}=C_{1,1,0}=C_{2,0,0}$

$=C_{2,0,1}=C_{1,1,1}=C_{4,0,0}=C_{2,0,2}=C_{0,2,1}$

$\therefore$ No sol.

Suppose $k\neq -b,-2b,-3b,-4b$.

$C_{0,0,4}=0$

$k=-4s$ or $C_{4,0,0}=0$

$C_{0,2,1}=0=C_{2,0,2}=C_{1,1,1}$

$C_{4,0,0}=-\frac{1}{4s}C_{2,0,1}$

$C_{2,0,1}=\frac{1}{s}C_{0,2,0}$

$C_{1,1,0}=-\frac{2s+k+b}{s}C_{0,2,0}$

if $k=-4s$: $0=C_{0,0,4}=C_{0,2,1}=C_{2,0,2}=C_{1,1,1}$

$C_{4,0,0}=-\frac{1}{4s^2}C_{0,2,0}$

$C_{2,0,1}=\frac{1}{s}C_{0,2,0}$

$C_{1,1,0}=\frac{2s-b}{s}C_{0,2,0}$

$=\frac{2-4s}{s}C_{0,2,0}$

$\Big\}\Rightarrow$ $b=6s-2$ or $C_{1,1,0}=C_{0,2,0}=0$

$C_{2,0,0}=\frac{(1-2s)(1-3s)-sr}{s^2}C_{0,2,0}$ $r=2s-1$ or

$$C_{2,0,1} = \frac{1}{s} C_{0,2,0}$$

$$C_{1,1,0} = -\frac{2s+k+b}{s} C_{0,2,0}$$

$$C_{1,1,0} = \frac{2+k}{s} C_{0,2,0}$$

$$C_{2,0,0} = \frac{(2+k)(s+1+k)-2sr}{2s^2} C_{0,2,0}$$

$$(2s+k)\, C_{2,0,0} = \frac{r(2+k)}{s} C_{0,2,0}$$

$$= \frac{2-4s}{s} C_{0,2,0} \qquad C_{1,1,0} = C_{0,2,0} = 0$$

$$C_{2,0,0} = \frac{(1-2s)(1-3s)-sr}{s^2} C_{0,2,0} = \frac{r(1-2s)}{s(-s)} C_{0,2,0} \Rightarrow r = 2s-1 \text{ or } s = \frac{1}{3}$$

if $C_{1,1,0} = C_{0,2,0} = 0$: No sol

else $b = 6s-2$:

if $r = 2s-1$: $C_{0,2,0} = C$ free

$C_{4,0,0} = -\frac{1}{4s^2} C$

$C_{2,0,1} = \frac{1}{s} C$

$C_{1,1,0} = \frac{2-4s}{s} C$

$C_{2,0,0} = \frac{(2s-1)^2}{s^2} C$

Let $C = \frac{1}{s}$, then

$$f(x,y,z) = -\frac{1}{4s}x^4 + x^2 z + \frac{(2s-1)^2}{s} x^2 + s y^2 + (2-4s)xy.$$

if $s = \frac{1}{3}$: $b = 0$, $k = -\frac{4}{3}$.

$C_{0,2,0} = C$ free

$C_{4,0,0} = -\frac{9}{4} C$, $C_{2,0,1} = 3C$

$C_{1,1,0} = 2C$, $C_{2,0,0} = -3r$

Let $C = \frac{1}{3}$, then

$$f(x,y,z) = -\frac{3}{4}x^4 + x^2 z - r x^2 + \frac{2}{3}xy + \frac{1}{3}y^2$$

else: $0 = C_{0,0,4} = C_{0,2,1} = C_{2,0,2} = C_{1,1,1} = C_{4,0,0} = C_{2,0,1} = C_{0,2,0} = C_{1,1,0} = C_{2,0,0}$

∴ No sol.

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