Conformal Mapping and its Application to Laplace's Equations

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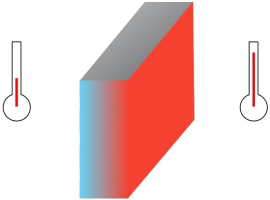
Motivation

Application of Laplace's equations:

- heat flow: steady-state temperature distribution



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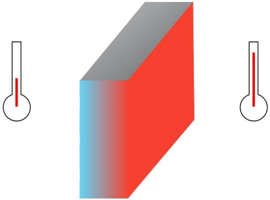
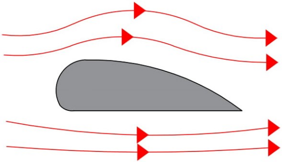


- aerodynamics: laminar flow over airfoils



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- Solving Laplace's equations on more complicated domains using **conformal mapping**

Where are we

Laplace's equation: introduction

A **partial differential equation (PDE)** is a differential equation involving partial derivatives with respect to more than one independent variable.

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Definition

Laplace's equation is the PDE of the form

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We will focus on **Dirichlet boundary conditions** in our case.

Laplace's equation: main techniques on simple domains

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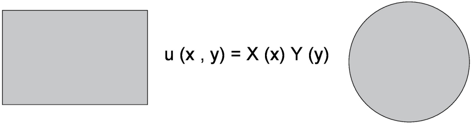
- Separation of variables



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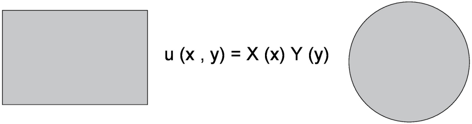

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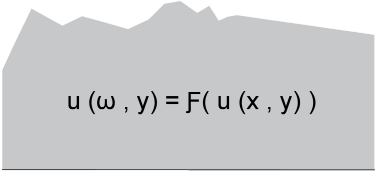

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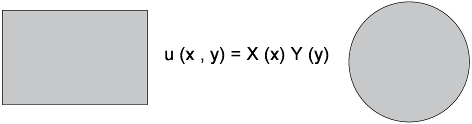

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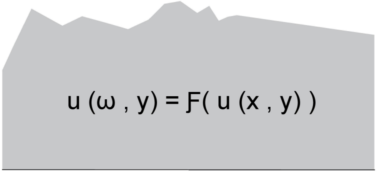
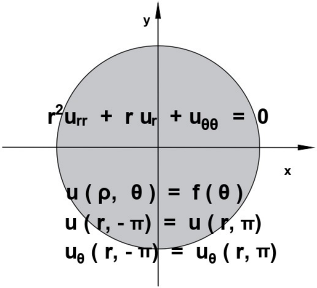


Next, we will have examples on circular domain and upper half-plane.

Examples: Separation of variables on a disk

Laplace's equation in the polar form: $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$



Separation of variables:

$$u(r, \theta) = R(r)\,\Theta(\theta)$$
$$\Longrightarrow \frac{r^2 R'' + r\,R'}{-R} = \frac{\Theta''}{\Theta} = \pm\lambda$$

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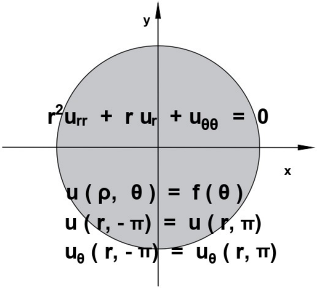

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STEP 1: **Eigenvalue problem**

$$\Theta''(\theta) = -\lambda\Theta$$

$$\Theta(-\pi) = \Theta(\pi)$$

$$\Theta'(-\pi) = \Theta'(\pi)$$

$$\lambda = n^2\,(n \in \mathbb{N}),\ \Theta_n(\theta) = \begin{cases} a_n \cos n\theta + b_n \sin n\theta & n = 1, 2, 3\ldots \\ \frac{a_0}{2} & n = 0 \end{cases}$$

STEP 2: Solving for $r^2 R'' + r\,R' - n^2 R = 0$

Guess $R(r) = r^p \implies R_n(r) = C_n r^n + D_n r^{-n}$

Well defined at $r = 0 \implies R_n(r) = C_n r^n \qquad (n = 0, 1, 2, 3, \ldots)$

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STEP 3: Combine two variables

By **superposition principle**,

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

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STEP 4: Apply boundary conditions and obtain the solution

$$u(\rho, \theta) = f(\theta) \implies f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^n \left(a_n \cos n\,\theta + b_n \sin n\,\theta\right)$$

By **Fourier series formula**, $\begin{cases} a_n = \frac{1}{\pi\,\rho^n} \int_{-\pi}^{\pi} f(\phi) \cos n\,\phi \, \mathrm{d}\phi \\ b_n = \frac{1}{\pi\,\rho^n} \int_{-\pi}^{\pi} f(\phi) \sin n\,\phi \, \mathrm{d}\phi \end{cases}$

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Applying **termwise integration**, **trigonometric identity** and **geometric series formula**, we obtain the solution

$$u(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \phi)} \, \mathrm{d}\phi$$

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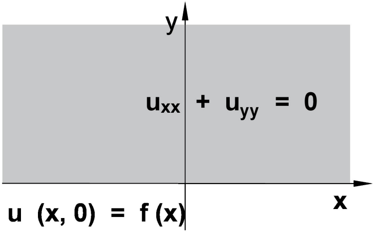
Remark

Note that this is the **Poisson Integral Formula** and

$$K(\theta = \frac{\rho^2 - r^2}{2\pi\left(\rho^2 + r^2 - 2\rho r\cos(\theta)\right)}$$

is the **Green's function** for Laplace's equation on a disk.

Examples: Fourier transform on upper half-plane



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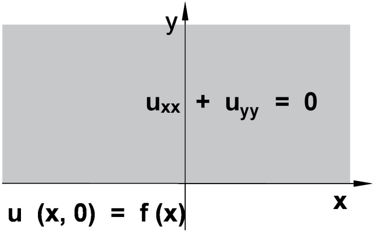

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Assumption:

- $f(x)$ and $u(x, y)$ can be decomposed by Fourier transform formula
- $\lim_{x \to -\infty} u(x, y) = 0 = \lim_{x \to \infty} u(x, y)$ and $\lim_{y \to \infty} u(x, y) = 0$

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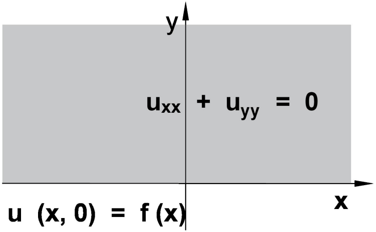

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Fourier transform with repect to x:

$$\hat{u}(\omega, y) = \int_{-\infty}^{\infty} u(x,y)\, e^{-i\omega x}\, \mathrm{d}x = \frac{1}{i\omega}\hat{u}_x(\omega, y)$$

Similarly, $\hat{u}_{xx}(\omega, y) = (i\omega)\hat{u}_x(\omega, y) = (i\omega)^2\hat{u}(\omega, y)$

STEP 1: Find a general solution:

Applying **Fourier transform formula** to $u_{xx} + u_{yy} = 0$ gives

$$(i\omega)^2 \hat{u}(\omega, y) + \hat{u}_{yy}(\omega, y) = 0$$

$$\Longrightarrow \ \hat{u}(\omega, y) = C_1(\omega)\, e^{\omega y} + C_2(\omega)\, e^{-\omega y}.$$

Since $u(x, y) \to 0$ as $y \to \infty$, it yields

$$\hat{u}(\omega, y) = C(\omega)\, e^{-|\omega| y} \qquad \omega \in (-\infty, \infty). \tag{1}$$

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STEP 2: Apply **boundary condition**:

Denote Boundary condition in Fourier form: $\hat{f}(\omega) = \mathcal{F}(f(x))$.

Then,

$$(1) \ \Longrightarrow \ \hat{u}(\omega, 0) = \hat{f}(\omega) = C(\omega)$$

$$\Longrightarrow \ \hat{u}(\omega, y) = \hat{f}(\omega)\, e^{-|\omega| y}$$

STEP 3: Apply the **inverse Fourier transform formula**

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-|\omega|x}\, e^{i\omega x}\, \mathrm{d}\omega$$

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Note that $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(\tau)\, e^{-i\omega\tau}\, \mathrm{d}\tau$, and by **Fubini's theorem**

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{0} e^{i\omega(x-\tau)+\omega y}\, \mathrm{d}\omega + \int_{0}^{\infty} e^{i\omega(x-\tau)-\omega y}\, \mathrm{d}\omega\right) \mathrm{d}\tau$$

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Remark

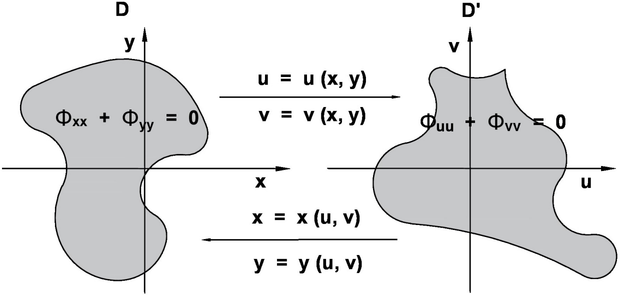
The **Green's function** on the upper half-plane is

$$K(x) = \frac{y}{\pi\left(y^2+x^2\right)}.$$

Where are we

Conformal mapping: invariance of Laplace's equations

Let's explore the property of an analytic map before introducing conformal map.



Theorem

Solutions of Laplace's equation on different domains are "preserved" under analytic map.

We shall prove it by the chain rule.

proof

Let $f : w(u, v) \to z(x, y)$ be analytic on domain D and $\Phi(x, y)$ be the harmonic function on D. Then Cauchy-Riemann equations are satisfied as follows:

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v} \tag{2}$$

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Applying the chain rule on ψ_{uu} yields

$$\Phi_{uu} = \frac{\partial}{\partial u}\left(\Phi_x \frac{\partial x}{\partial u} + \Phi_y \frac{\partial y}{\partial u}\right)$$

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Applying the chain rule on ψ_{uu} yields

$$\begin{aligned}\Phi_{uu} &= \frac{\partial}{\partial u}\left(\Phi_x \frac{\partial x}{\partial u} + \Phi_y \frac{\partial y}{\partial u}\right)\\ &= \Phi_{xx}\left(\frac{\partial x}{\partial u}\right)^2 + \Phi_{yy}\left(\frac{\partial y}{\partial u}\right)^2\\ &+ 2\,\Phi_{xy}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}\right) + \Phi_x \frac{\partial^2 x}{\partial u^2} + \Phi_y \frac{\partial^2 y}{\partial u^2},\end{aligned}$$

similarly,

$$\Phi_{vv} = \Phi_{xx} \left(\frac{\partial x}{\partial v} \right)^2 + \Phi_{yy} \left(\frac{\partial y}{\partial v} \right) + 2\,\Phi_{xy} \left(\frac{\partial x}{\partial v} \frac{\partial y}{\partial v} \right) + \Phi_x \frac{\partial^2 x}{\partial v^2} + \Phi_y \frac{\partial^2 y}{\partial v^2}.$$

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Combining equations 2 and 3, we have

$$\Phi_{uu} + \Phi_{vv} = \Phi_{xx}\left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2\right] + \Phi_{yy}\left[\left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2\right] + 2\,\Phi_{xy}\left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial v} + \frac{\partial x}{\partial u}\frac{\partial y}{\partial u}\right) + \Phi_x\left(\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2}\right) + \Phi_y\left(\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2}\right)$$

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$$+ \Phi_x\left(\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2}\right) + \Phi_y\left(\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2}\right)$$
$$= |\nabla x(u,v)|^2\ (\Phi_{xx} + \Phi_{yy}) = 0.$$

Conformal mapping: introduction

Definition

A function f on $\mathbb{C}$ is conformal

if it preserves angles locally.

[1] Daniel R. Strebe, 15 December 2011

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Theorem

An analytic function f is conformal at z_0 if its derivative $f'(z_0) \neq 0$.

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Figure: Mercator projection[1]

We will introduce two mappings: Möbius transformation and Schwarz-Christoffel transformation.

[1] Daniel R. Strebe, 15 December 2011

Conformal mapping: Möbius transform

Definition

Möbius transform is a complex-valued function in form of

$$w = f(z) = \frac{az + b}{cz + d}$$

where a, b, c and d are complex constants with $ad \neq bc$.

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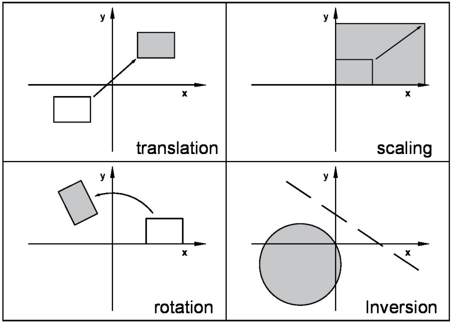

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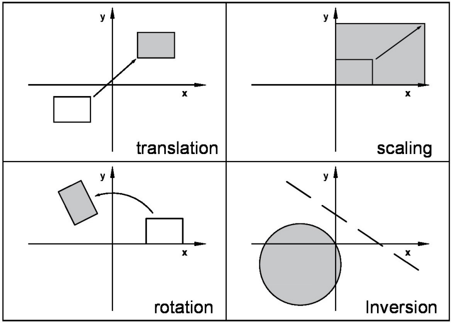
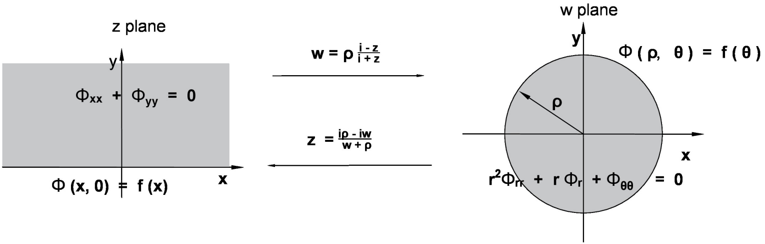


Theorem

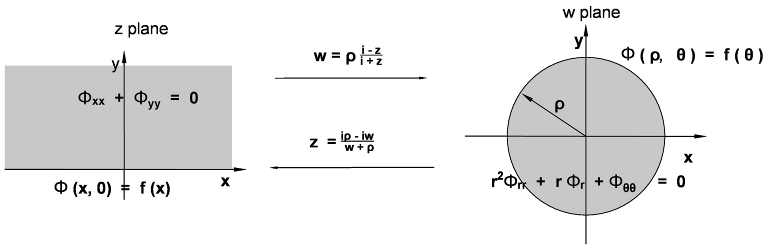
Möbius transform is a conformal map.

Example: upper half-plane to disk



Mapping function: $w(u, v) = \rho \dfrac{i - z}{i + z} \leftrightarrow z(x, y) = \dfrac{i\,\rho - i\,w}{w + \rho}$

Example: upper half-plane to disk



$$\text{Mapping function: } w(u,v) = \rho\frac{i-z}{i+z} \leftrightarrow z(x,y) = \frac{i\,\rho - i\,w}{w+\rho}$$

$$\Longrightarrow \begin{cases} x = \dfrac{2\rho r\ \sin\theta}{\rho^2 + r^2 + 2\rho r\ \cos\theta} \\ y = \dfrac{\rho^2 - r^2}{\rho^2 + r^2 + 2\rho r\ \cos\theta} \end{cases}$$

Recall the formula of solution on upper half-plane:

$$\Phi(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\tau) \frac{y}{y^2 + (x - \tau)^2} \, \mathrm{d}\tau$$

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Thus, replacing x , y with r and θ gives us

$$\Phi(r, \theta) = \frac{1}{\pi} \int_{\tau(-\infty)}^{\tau(\infty)} f(\phi(\tau)) \frac{y(r, \theta)}{y^2(r, \theta) + (x(r, \theta) - \tau(\phi))^2} \, \mathrm{d}(\tau(\phi))$$

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$$\begin{aligned}\Phi(r,\theta) &= \frac{1}{\pi}\int_{\tau(-\infty)}^{\tau(\infty)} f(\phi(\tau))\frac{y(r,\theta)}{y^2(r,\theta)+(x(r,\theta)-\tau(\phi))^2}\,\mathrm{d}(\tau(\phi))\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\frac{\rho^2-r^2}{\rho^2+r^2-2\rho r\,\cos(\theta-\phi)}\,\mathrm{d}\phi,\end{aligned}$$

where $\tau(\phi)$ is derived from $x(\rho,\theta)$ and $f(\phi)$ is the boundary condition at $r=\rho$ on the disk.

Conformal mapping: Schwarz-Christoffel transformation

SC Mapping Theorem[2]

A one-to-one conformal function that maps the upper-half plane onto the polygon is

$$f(z) = A\int_0^z (\zeta - x_1)^{\frac{\theta_1}{\pi}} (\zeta - x_2)^{\frac{\theta_2}{\pi}} \ldots (\zeta - x_{n-1})^{\frac{\theta_{n-1}}{\pi}}\, \mathrm{d}\zeta + B.$$

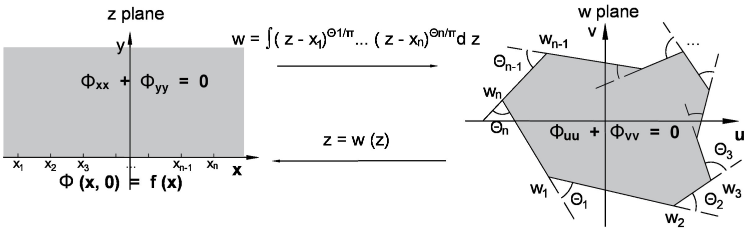
[2] Edward B Saff/Arthur David Snider: Fundamentals of complex analysis for mathematics, science, and engineering, 1976.

Conformal mapping: Schwarz-Christoffel transformation

SC Mapping Theorem[2]

A one-to-one conformal function that maps the upper-half plane onto the polygon is

$$f(z) = A\int_0^z (\zeta - x_1)^{\frac{\theta_1}{\pi}}(\zeta - x_2)^{\frac{\theta_2}{\pi}} \dots (\zeta - x_{n-1})^{\frac{\theta_{n-1}}{\pi}}\, \mathrm{d}\zeta + B.$$



[2]Saff/Snider: Fundamentals of complex analysis for mathematics, science, and engineering (see n. 2).

Example: flow over a corner

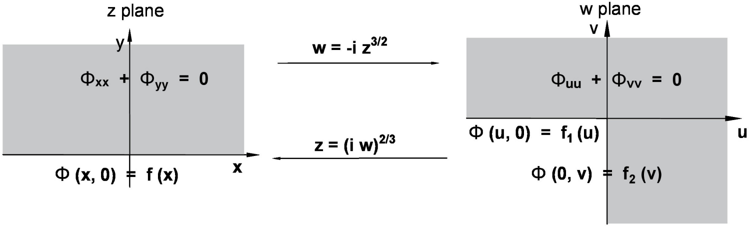


Figure: From upper half-plane to the plane excluding the third quadrant

STEP 1: Find the **mapping function**

By SC mapping formula, we have

$$w = -i\, z^{\frac{3}{2}} = -i\, |z|^{\frac{3}{2}}\, e^{i\frac{3}{2}(\mathrm{Arg}\, z)} \leftrightarrow$$

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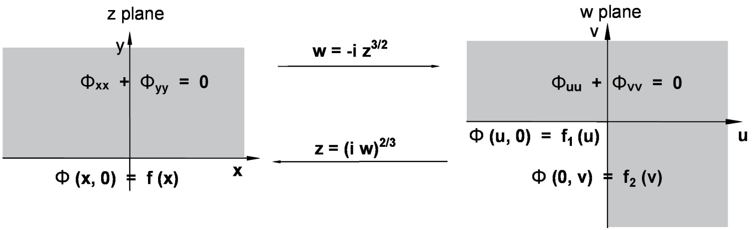

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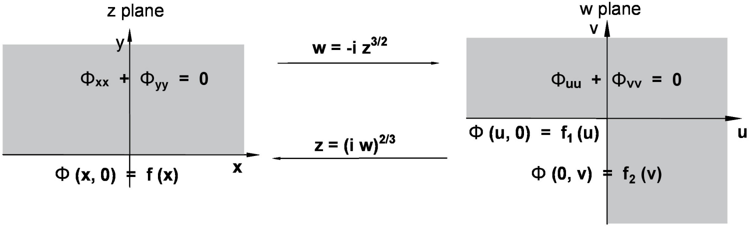

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$$\Longrightarrow \begin{cases} x = (u^2 + v^2)^{\frac{1}{3}} \cos(\frac{2}{3} \operatorname{Arg} w + \frac{\pi}{3}) \\ y = (u^2 + v^2)^{\frac{1}{3}} \sin(\frac{2}{3} \operatorname{Arg} w + \frac{\pi}{3}) \end{cases}$$

Recall the formula of solution to the upper half-plane:

$$\Phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{y^2 + (x - \tau)^2 \, \mathrm{d}\tau}$$

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$$\Phi(u,v) = \frac{2y}{3\pi}\int_{-\infty}^{0} \Big(\frac{(-t)^{\frac{1}{3}}\, f_1(t)}{y^2 + (x + (-t)^{\frac{2}{3}})^2} + \int_{-\infty}^{0} \frac{(-t)^{\frac{1}{3}}\, f_2(t)}{y^2 + (x - (-t)^{\frac{2}{3}})^2} \Big)\,\mathrm{d}t$$

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where $x(u, v) = \left((u^2 + v^2)^{\frac{1}{3}} \cos\left(\frac{2}{3} \operatorname{Arg} w + \frac{\pi}{3}\right) \right)^2$,

$y(u, v) = \left((u^2 + v^2)^{\frac{1}{3}} \sin\left(\frac{2}{3} \operatorname{Arg} w + \frac{\pi}{3}\right) + (-t)^{\frac{2}{3}} \right)^2$ and

$$t = \begin{cases} -(-\tau)^{\frac{3}{2}} & \tau < 0 \\ -\tau^{\frac{3}{2}} & \tau > 0 \end{cases}.$$

Conclusion

What we have learned or reviewed:

- application of Laplace's equation

What I have realized:

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- solving Laplace's equations by two main techniques

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- conformal mapping can solve some complicated domains

Conclusion

What we have learned or reviewed:

- application of Laplace's equation
- solving Laplace's equations by two main techniques
- introduction to conformal mapping

What I have realized:

- conformal mapping can solve some complicated domains
- SC map is difficult to apply, considering the complicated integration and inverse map

UNBC

Questions

Thanks for your attendance.

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Any questions welcome!

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Short questions? Ask me now!

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One of my favorite quotes

"The one who knows all the answers has not been asked all the questions."

– Confucius

References

- Lan, Weixian: Conformal mapping and its application in Laplace's equation, in: 2019.
- Saff, Edward B and Arthur David Snider: Fundamentals of complex analysis for mathematics, science, and engineering, 1976.