Conformal Mapping and its Application to Laplace's Equation

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Undergraduate Summer Project

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Acknowledgement

This is an undergraduate project in 2019 summer, yet my first one in mathematics. It covers topics widely in complex analysis and differential equations, which fall into my broad range of interests. In this project, all the solutions were derived step by step with detailed computing process, but could not be justified rigorously for my lack of background in analysis. I learned to perform algebraic calculation in a good manner, apply old techniques to solve new problems, and put my thoughts in words. This is another way of study as opposed to waiting for solutions in class.

The project was supervised by Dr. Andy Wan. Thanks for your time and guidance.

The followings are my opinions about the research project:

- Most solutions really cannot be expressed by elementary functions; instead they are in a more complicated form beyond my understanding.
- To get to know a new concept cannot be done by Wikipedia in minutes. The description of the concept can include another new concept which you would also want to look up and so forth.
- Latex is easy to pick up, but hard to master. Overall, it is complicated.
- Even quick notes on scratch papers have to be legible; otherwise, scribble does not give any mathematical intuition but annoys yourself.
- Doing calculation in head can lead to unnecessary mistakes, so I encourage doing that by hand if time permits. It is helpful in exams, though.
- Research is a special job, because researchers are one of the few who can claim they really love their jobs.

Contents

1 Introduction

This project involves applying conformal mappings to solving Laplace's equation. We assume the readers have sufficient knowledge in Laplace's equations, Fourier series, Fourier transform and conformal mapping. For better understanding, please refer to **Math 230** lecture notes for Laplace's equations and Fourier series and **Math 201** textbook for conformal mapping. In section 2, we suggest that an equivalent domain can preserve the solution to Laplace's equations. In section 3, we introduce the main techniques for solving Laplace's equation on some simple domains: disk, annulus, upper half-plane and semi-infinite stripe. As we are familiarized with solutions on those particular domains, we will apply conformal mapping to transform more irregular domains to one of the simple ones to derive the solution.

2 Invariance of Solutions to Laplace's Equations

In this section, we will introduce a key theorem, relating solutions of Laplace's equations on different domains.

Theorem 1. *If two domains D and D' are related by an analytic and one-to-one function $f(u,v) = x(u,v) + iy(u,v)$, then their Laplacian $\nabla^2\psi(u,v)$ on D and $\nabla^2\phi(x,y)$ on D' are related by*

$$\nabla^2\psi(u,v) = |\nabla x(u,v)|^2 \nabla^2\phi(x,y),$$

where $\nabla^2\psi(u,v) = \nabla^2\phi(x(u,v), y(u,v))$.

Proof. Since $f(u,v) = x(u,v) + iy(u,v)$ is analytic, Cauchy-Riemann equations are satisfied as follows:

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v} \tag{1}$$

$$\frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u} \tag{2}$$

Applying the chain rule on ψ_{uu} yields

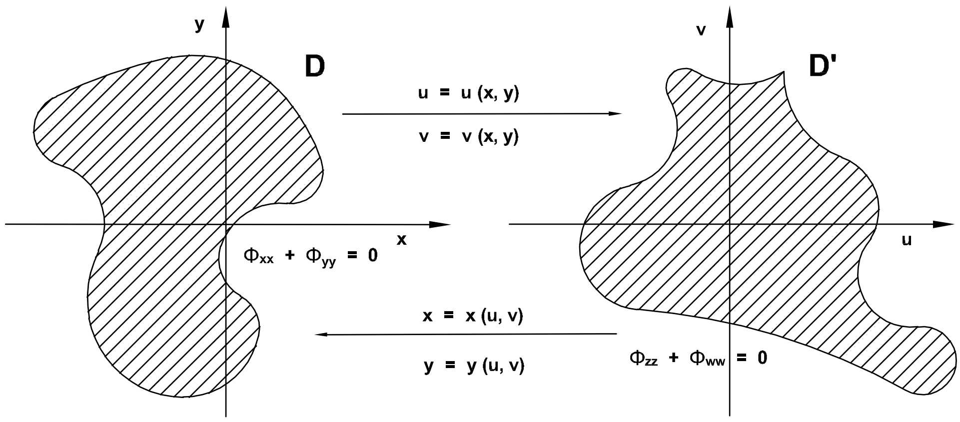

Figure 1: Invariance of Laplace's equations

$$\begin{aligned}\psi_{uu} &= \frac{\partial}{\partial u}\,(\phi_x \frac{\partial x}{\partial u} + \phi_y \frac{\partial y}{\partial u}) \\ &= \frac{\partial \phi_x}{\partial u}\frac{\partial x}{\partial u} + \phi_x \frac{\partial^2 x}{\partial u^2} + \frac{\partial \phi_y}{\partial u}\frac{\partial y}{\partial u} + \phi_y \frac{\partial^2 y}{\partial u^2} \\ &= \phi_{xx}\left(\frac{\partial x}{\partial u}\right)^2 + \phi_{yy}\left(\frac{\partial y}{\partial u}\right)^2 + 2\phi_{xy}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}\right) + \phi_x \frac{\partial^2 x}{\partial u^2} + \phi_y \frac{\partial^2 y}{\partial u^2},\end{aligned}$$

likewise for ψ_{vv},

$$\psi_{vv} = \phi_{xx}\left(\frac{\partial x}{\partial v}\right)^2 + \phi_{yy}\left(\frac{\partial y}{\partial v}\right) + 2\phi_{xy}\left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial v}\right) + \phi_x \frac{\partial^2 x}{\partial v^2} + \phi_y \frac{\partial^2 y}{\partial v^2}.$$

Combining equations 1 and 2, we have

$$\begin{aligned}\psi_{uu} + \psi_{vv} = \phi_{xx}&\left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2\right] + \phi_{yy}\left[\left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2\right] + 2\phi_{xy}\left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial v} + \frac{\partial x}{\partial u}\frac{\partial y}{\partial u}\right) \\ &+ \phi_x\left(\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2}\right) + \phi_y\left(\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2}\right) \\ &= \phi_{xx}\left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2\right] + \phi_{yy}\left[\left(-\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial u}\right)^2\right] + 2\phi_{xy}\left(\frac{\partial x}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\right) \\ &= (\phi_{xx} + \phi_{yy})\left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2\right] \\ &= |\nabla x(u,v)|^2\;(\phi_{xx} + \phi_{yy}).\end{aligned}$$

Therefore, we justified the theorem $\nabla^2\psi(u,v) = |\,\nabla x(u,v)\,|^2\;\nabla^2\phi(x,y)$. □

Recall that a function $\phi(x,y)$ is harmonic if $\nabla^2\phi(x,y) = \phi_{xx} + \phi_{yy} = 0$. This leads to the corollary next.

Corollary. *If $\phi(x,y)$ is harmonic, then $\psi(u,v)$ is also harmonic.*

Proof. Since $\phi(x,y)$ is harmonic, it follows from Theorem 1 that

$$\begin{aligned}\nabla^2\psi(u,v) &= |\nabla x(u,v)|^2\,(\phi_{xx}+\phi_{yy})\\ &= |\nabla x(u,v)|^2 \times 0\\ &= 0.\end{aligned}$$

Therefore, $\psi(u,v)$ is harmonic. □

3 Solutions of Laplace's Equations on Simple Domains

We will focus on solving Laplace's equations on simple domains: disk, annulus, upper-half plane and semi-infinite stripe. We consider Dirichlet boundary conditions, unless specified otherwise. Two main techniques are separation of variables and Fourier transform.

3.1 Circular Domain

This section is for solving Laplace's equations with Dirichlet and Neumann BC[1] on circular domain. We will use the method of separating variables.

3.1.1 Dirichlet boundary condition

Consider the following boundary value problem:

$$r^2u_{rr}+ru_r+u_{\theta\theta}=0$$

$$u(\rho,\theta)=f(\theta)$$

$$u(r,-\pi)=u(r,\pi)$$

$$u_\theta(r,-\pi)=u_\theta(r,\pi)$$

[1]BC stands for boundary conditions

where $f(\theta)$ is at least piecewise continuous.

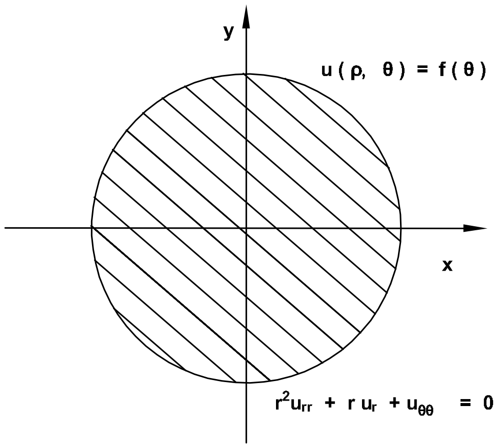

Figure 2: Dirichlet boundary condition on the circle centered at the origin

The solution is

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\frac{r^2-\rho^2}{r^2+\rho^2-2\rho r\cos(\theta-\phi)}\,\mathrm{d}\phi.$$

We derive the solution as follows.

We guess the form of $u(r,\theta) = R(r)\,\Theta(\theta)$. Then,

$$r^2R'' + rR' + R\Theta'' = 0\,,$$

$$\frac{r^2R'' + rR'}{-R} = \frac{\Theta''}{\Theta}\,.$$

Notice that the only possible case is

$$\frac{r^2R'' + rR'}{-R} = \frac{\Theta''}{\Theta} = \pm\lambda,$$

where λ is a non-negative constant with the sign and value to be determined.

If we rearrange the equations, we get

$$r^2R'' + rR' \pm \lambda R = 0, \tag{3}$$

$$\Theta'' = \pm\lambda\Theta. \tag{4}$$

From equation 4, we have the following boundary problem for nontrivial solutions:

$$\Theta'' = \pm\lambda\Theta$$

$$\Theta(-\pi) = \Theta(\pi)$$

$$\Theta'(-\pi) = \Theta'(\pi)$$

We will approach the above as the eigenvalue problems with eigenvalue λ and eigenfunction $\Theta(\theta)$ (details in Appendix A), where there is a solution if and only if $\Theta'' = -\lambda\Theta$ $(\lambda \geqslant 0)$.

$$\lambda = n^2 \qquad n = 0, 1, 2\ldots$$

$$\Theta_n(\theta) = \begin{cases} a_n \cos n\theta + b_n \sin_n \theta & n > 0 \\ \frac{a_0}{2} & n = 0 \end{cases}$$

From equation 3, we have

$$r^2 R'' + rR' - \lambda R = 0.$$

Take $R(r) = r^p$, where p is constant, we have

$$p(p-1)r^p + pr^p - \lambda r^p = 0,$$

$$p^2 - \lambda = 0,$$

$$p = \pm\sqrt{\lambda}.$$

Since $\lambda = n^2$, $p = \pm n$. Therefore, $R_n(r) = C_1 r^n + C_2 r^{-n}$. However, in order for $R(r)$ to be well defined at $r = 0$, the term $C_2 r^{-n}$ must vanish. Hence, it suggests

$$R_n(r) = \begin{cases} r^n & n > 0, \\ 1 & n = 0. \end{cases}$$

Therefore, by principle of superposition,

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta))r^n.$$

Applying the boundary condition $u(\rho,\theta) = f(\theta)$, we obtain

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta))\rho^n.$$

By Fourier series formula, we find the values of coefficients

$$a_n = \frac{1}{\pi \rho^n} \int_{-\pi}^{\pi} f(\phi) \cos n\phi \, \mathrm{d}\phi,$$

$$b_n = \frac{1}{\pi \rho^n} \int_{-\pi}^{\pi} f(\phi) \sin n\phi \, \mathrm{d}\phi.$$

Assuming $f(\phi)$ is at least piecewise continuous on $(-\pi, \pi)$, then termwise integration can be applied, which leads to, after using angle-sum identity,

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \, \mathrm{d}\phi + \sum_{n=1}^{\infty} \left(\frac{r^n}{\pi \rho^n} \int_{-\pi}^{\pi} f(\phi)(\cos n\theta \cos n\phi + \sin n\theta \sin n\phi) \, \mathrm{d}\phi \right) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \left(1 + 2 \sum_{n=1}^{\infty} \frac{r^n \cos n(\phi - \theta)}{\rho^n} \right) \mathrm{d}\phi \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \left(1 + \sum_{n=1}^{\infty} \frac{r^n (e^{in(\phi-\theta)} + e^{-in(\phi-\theta)})}{\rho^n} \right) \mathrm{d}\phi \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \left(1 + \frac{re^{i(\theta-\phi)}}{\rho - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{\rho - re^{-i(\theta-\phi)}} \right) \mathrm{d}\phi \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \left(\frac{\rho^2 - r^2}{r^2 + \rho^2 - 2\rho r \cos(\theta - \phi)} \right) \mathrm{d}\phi .
\end{aligned}$$

□

Remark on periodic boundary condition

One may be curious whether it matters if the boundary condition is prescribed on $[0, 2\pi]$. It turns out that the solution is independent of the chosen domain, ie, $[0, 2\pi]$ and $[-\pi, \pi]$ are expected to yield a same result.

Specifically, we can consider an arbitrary function $F(\theta)$ defined on $\mathbb{R}$ with a period of 2π. The function is defined as follows:

$$F(\theta) = \begin{cases} f_1(\theta), & [-\pi, \tau), \\ f_2(\theta), & [\tau, \pi), \\ f_2(\theta - 2k\pi), & [2k\pi + \tau, 2k\pi + \pi), \\ f_1(\theta - (2k+2)\pi), & [2k\pi + \pi, (2k+2)\pi + \tau), \end{cases}$$

where $\tau \in [-\pi, \pi)$ and $k \in \mathbb{Z}$.

If the chosen domain is $[-\pi, \pi]$, functions $f_1(\theta)$ and $f_2(\theta)$ will be evaluated on $[-\pi, \tau)$ and $[\tau, \pi)$, respectively. Hence, we can write

$$\int_{-\pi}^{\pi} F(\theta)\,\mathrm{d}\theta = \int_{-\pi}^{\tau} f_1(\theta)\,\mathrm{d}\theta + \int_{\tau}^{\pi} f_2(\theta)\,\mathrm{d}\theta.$$

If the chosen domain is defined on $[2k\pi+\tau,\, 2(k+1)\pi+\tau)$, functions $f_2(\theta-2k\pi)$ and $f_1(\theta-(2k+2)\pi)$ will be evaluated on $[2k\pi+\tau,\, 2k\pi+\pi)$ and $[2k\pi+\pi,\, 2(k+1)\pi+\tau)$. In the latter case, if replacing the variable $(\theta-2k\pi)$ with α and $(\theta-(2k+2)\pi)$ with β, then

$$\begin{aligned}\int_{2k\pi+\tau}^{(2k+2)\pi+\tau} F(\theta)\,\mathrm{d}\theta &= \int_{2k\pi+\tau}^{2k\pi+\pi} f_2(\theta-2k\pi)\,\mathrm{d}\theta + \int_{2k\pi+\pi}^{(2k+2)\pi+\tau} f_1(\theta-(2k+2)\pi)\,\mathrm{d}\theta \\ &= \int_{\tau}^{\pi} f_2(\alpha)\,\mathrm{d}\alpha + \int_{-\pi}^{\tau} f_1(\beta)\,\mathrm{d}\beta.\end{aligned}$$

From the above, two integrations will yield the same result. Since τ and k are arbitrarily selected, any general case can be represented as the above form.

3.1.2 Neumann boundary condition

Given the boundary condition problem:

$$\begin{cases} r^2u_{rr}+ru_r+u_{\theta\theta}=0 \\ u_r(\rho,\theta)=f(\theta) \\ u(r,-\pi)=u(r,\pi) \\ u_\theta(r,-\pi)=u_\theta(r,\pi) \end{cases}$$

where $f(\theta)$ is at least piecewise continuous.
The solution is

$$u(r,\theta) = -\frac{\rho}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,\ln(r^2+\rho^2-2\rho r\cos(\theta-\phi))\,\mathrm{d}\phi + C,$$

where C is constant.

We will apply the same method as we did in the case of Dirichlet boundary condition and get the general solution of the form,

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(n\theta)+b_n\sin(n\theta))r^n.$$

Assuming termwise differentiation holds, we differentiate the equation with respect to r to obtain

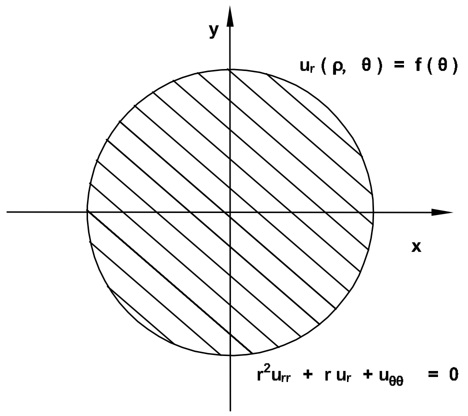

Figure 3: Neumann boundary condition on the circle centered at the origin

$$u_r(\rho,\theta) = \sum_{n=1}^{\infty} (na_n r^{n-1}\cos(n\theta) + nb_n r^{n-1}\sin(n\theta)), \tag{5}$$

and apply the boundary condition $u_r(\rho,\theta) = f(\theta)$, then

$$f(\theta) = \sum_{n=1}^{\infty} (na_n \rho^{n-1}\cos(n\theta) + nb_n \rho^{n-1}\sin(n\theta)).$$

Determining the coefficients using the Fourier series formula gives:

$$na_n\rho^{n-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\cos(n\phi)\,\mathrm{d}\phi,$$
$$nb_n\rho^{n-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\sin(n\phi)\,\mathrm{d}\phi.$$

Plug in the coefficients to equation 5, assuming the conditions for termwise integration are satisfied, then we have

$$\begin{aligned}
u_r(r,\theta) &= \sum_{n=1}^{\infty} \frac{r^{n-1}}{\pi\rho^{n-1}} \int_{-\pi}^{\pi} f(\phi)\,(\cos n\phi \cos n\theta + \sin n\phi \sin n\theta)\,\mathrm{d}\phi \\
&= \sum_{n=1}^{\infty} \frac{r^{n-1}}{\pi\rho^{n-1}} \int_{-\pi}^{\pi} f(\phi) \cos n(\phi-\theta)\,\mathrm{d}\phi \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \sum_{n=1}^{\infty} \left(\frac{r}{\rho}\right)^{n-1} \cos n(\phi-\theta)\,\mathrm{d}\phi \\
&= \frac{\rho}{\pi r} \int_{-\pi}^{\pi} f(\phi) \left(\sum_{n=1}^{\infty} \frac{r^n (e^{in(\phi-\theta)} + e^{-in(\phi-\theta)})}{2\rho^n} \right) \mathrm{d}\phi \\
&= \frac{\rho}{2\pi r} \int_{-\pi}^{\pi} f(\phi) \left(\frac{re^{i(\theta-\phi)}}{\rho - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{\rho - re^{-i(\theta-\phi)}} \right) \mathrm{d}\phi \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \frac{\rho^2 \cos(\phi-\theta) - \rho r}{r^2 + \rho^2 - 2\rho r \cos(\phi-\theta)}\,\mathrm{d}\phi.
\end{aligned}$$

Integrating $u_r(r,\theta)$ with respect to r gives

$$\begin{aligned}
u(r,\theta) &= \int u_r(r,\theta)\,\mathrm{d}r \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \int_0^r \frac{\rho^2 \cos(\phi-\theta) - \rho s}{s^2 + \rho^2 - 2\rho s \cos(\phi-\theta)}\,\mathrm{d}s\,\mathrm{d}\phi \quad \text{(Reverse the order of integration)} \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \left[-\frac{\rho}{2} \ln(r^2 + \rho^2 - 2\rho r \cos(\phi-\theta)) + C(\phi,\theta) \right] \mathrm{d}\phi \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) C(\phi,\theta)\,\mathrm{d}\phi - \frac{\rho}{2\pi} \int_{-\pi}^{\pi} f(\phi) \ln(r^2 + \rho^2 - 2\rho r \cos(\phi-\theta))\,\mathrm{d}\phi, \qquad (6)
\end{aligned}$$

where $C(\phi,\theta)$ is some function with variables ϕ and/or θ.

We can write equation 6 as a linear combination of homogeneous and particular solutions:

$$u(r,\theta) = u_h(\theta) + u_p(r,\theta),$$

where $u_h(\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi)\, C(\phi,\theta)\,\mathrm{d}\phi$ and $u_p(r,\theta) = -\frac{\rho}{2\pi} \int_{-\pi}^{\pi} f(\phi)\, \ln(r^2 + \rho^2 - 2\rho r \cos(\phi-\theta))\,\mathrm{d}\phi$.

To determine the form of $u_h(\theta)$, we go back to the boundary condition:

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0 \tag{7}$$

$$u_r(\rho,\theta) = f(\theta) \tag{8}$$

$$u(r,-\pi) = u(r,\pi) \tag{9}$$

$$u_\theta(r,-\pi) = u_\theta(r,\pi) \tag{10}$$

We know that the particular solution $u_p(r,\theta)$ alone can satisfy all the conditions 7, 8, 9 and 10 by setting $u_h(\theta) = 0$.

Therefore, $u_h(\theta)$ must satisfy the following conditions as well:

$$u_h''(\theta) = 0$$

$$u_h(-\pi) = u_h(\pi)$$

$$u_h'(-\pi) = u_h'(\pi)$$

Then, it gives us $u_h(\theta) = C$, where C is constant.

In conclusion, the solution is in the form of

$$u(r,\theta) = -\frac{\rho}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,\ln(r^2+\rho^2-2\rho r\cos(\theta-\phi))\,\mathrm{d}\phi + C\,.$$

□

3.2 Annular Domain

In this subsection, we will solve the Laplace's equation on the annular domain. The solution on an annulus is important, because it has the potential to solve Laplace's equation on more complicated domains, under conformal mapping, which we will discuss in Section 4.

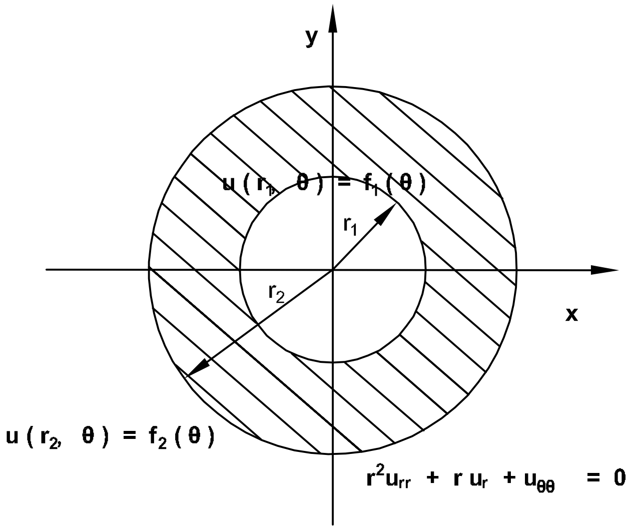

Figure 4:

Figure 5: Dirichlet condition on the annular domain

Laplace's equation with Dirichlet boundary conditions for the annular domain is

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0\,,$$

$$u(r_1,\theta) = f_1(\theta)\,,$$

$$u(r_2,\theta) = f_2(\theta)\,,$$

$$u(r,-\pi) = u(r,\pi)\,,$$

$$u'(r,-\pi) = u'(r,\pi)\,,$$

where $0 < r_1 < r_2$.

We can still apply the method separation of variables. For the Dirichlet boundary conditions, we obtain

$$R(r_1)\,\Theta(\theta) = f_1(\theta)\,,$$

$$R(r_2)\,\Theta(\theta) = f_2(\theta)\,.$$

In general, $f_1(\theta)$ and $f_2(\theta)$ are not a multiple of each other. Therefore, we should not expect one particular solution to satisfy both of the boundary conditions. Alternatively, we can find two particular solutions separately, one of which, say u_{p_1}, satisfies

$$R(r_1)\,\Theta(\theta) = f_1(\theta),$$

$$R(r_2)\,\Theta(\theta) = 0.$$

and the other, say u_{p_2}, satisfies

$$R(r_1)\,\Theta(\theta) = 0,$$

$$R(r_2)\,\Theta(\theta) = f_2(\theta).$$

With this main idea, we shall begin solving the equations.

By separating variables, we replace $u(r,\theta)$ with $R(r)\Theta(\theta)$, and it can be derived from the Laplace's equation that

$$\frac{r^2R''+rR'}{R} = -\frac{\Theta''}{\theta} = \lambda.$$

From our previous practice, it is clear that $\lambda \geq 0$, and we can solve for $\Theta(\theta)$:

$$\begin{cases} \Theta'' = -\lambda\theta \\ \Theta(-\pi) = \Theta(\pi) \\ \Theta'(-\pi) = \Theta(\pi) \end{cases}$$

gives

$$\lambda = n^2, \qquad \Theta_n(\theta) = \begin{cases} C & n = 0, \\ a_n\cos n\theta + b_n\sin n\theta & n = 1,2,3\dots. \end{cases} \tag{11}$$

We now start the case for the first particular solution u_{p_1}. We have the equations for $R(r)$

$$r^2R''+rR'-n^2R = 0$$

$$R(r_2) = 0.$$

If $n = 0$, we solve $r^2R''+rR' = 0$ for

$$R_0 = C_1\ln r + C_2.$$

If $n = 1,2,3\dots$, we shall obtain

$$R_n = d_nr^n + e_nr^{-n}.$$

We apply the boundary condition $R(r_2) = 0$, and it follows

$$\begin{cases} C_1\ln r_2 + C_2 = 0 \\ d_nr_2^n + e_nr_2^{-n} = 0 \end{cases}$$

and

$$\begin{cases} C_2 = -C_1 \ln r_2 \\ e_n = -d_n r_2^{2n}. \end{cases}$$

Therefore,

$$R_n(r) = \begin{cases} C_1(\ln r - \ln r_2) & n = 0, \\ d_n(r^n - r_2^{2n} r^{-n}) & n = 1, 2, 3 \dots. \end{cases} \tag{12}$$

Combining equations 11 and 12, we obtain

$$u_{p_1} = C_1(\ln r - \ln r_2) + \sum_{n=1}^{\infty} d_n(r^n - r_2^{2n} r^{-n})(a_n \cos n\theta + b_n \sin n\theta).$$

Applying boundary condition $u_{p_1}(r_1, \theta) = f_1(\theta)$, we get

$$f_1(\theta) = C_1 \ln \frac{r_1}{r_2} + \sum_{n=1}^{\infty} d_n(r_1^n - r_2^{2n} r_1^{-n})(a_n \cos n\theta + b_n \sin n\theta).$$

Use the formula of Fourier series, we have

$$C_1 \ln \frac{r_1}{r_2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(\theta)\, \mathrm{d}\theta,$$

$$d_n a_n (r_1^n - r_2^{2n} r_1^{-n}) = \frac{1}{\pi} \int_{-\pi}^{\pi} f_1(\theta) \cos n\theta\, \mathrm{d}\theta,$$

$$d_n b_n (r_1^n - r_2^{2n} r_1^{-n}) = \frac{1}{\pi} \int_{-\pi}^{\pi} f_1(\theta) \sin n\theta\, \mathrm{d}\theta.$$

Hence, the first particular solution is

$$\begin{aligned} u_{p_1}(r, \theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(\phi) \frac{\ln r/r_2}{\ln r_1/r_2}\, \mathrm{d}\phi + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r^n - r_2^{2n} r^{-n}}{r_1^n - r_2^{2n} r_1^{-n}} \int_{-\pi}^{\pi} f_1(\phi) \cos n(\theta - \phi)\, \mathrm{d}\phi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(\phi) \left(\frac{\ln r/r_2}{\ln r_1/r_2} + 2 \sum_{n=1}^{\infty} \frac{r^n - r_2^{2n} r^{-n}}{r_1^n - r_2^{2n} r_1^{-n}} \cos n(\theta - \phi) \right) \mathrm{d}\phi. \end{aligned}$$

For the second particular solution u_{p_2}, we have the equations for $R(r)$

$$r^2 R'' + r R' - n^2 R = 0$$

$$R(r_1) = 0.$$

We then follow the same steps and finally reach

$$u_{p_2}(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_2(\phi) \left(\frac{\ln r/r_1}{\ln r_2/r_1} + 2 \sum_{n=1}^{\infty} \frac{r^n - r_1^{2n} r^{-n}}{r_2^n - r_1^{2n} r_2^{-n}} \cos n(\theta - \phi) \right) \mathrm{d}\phi.$$

As stated at the beginning, the full solution is the sum of the two particular solutions

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(\phi)\left(\frac{\ln r/r_2}{\ln r_1/r_2} + 2\sum_{n=1}^{\infty}\frac{r^n - r_2^{2n}r^{-n}}{r_1^n - r_2^{2n}r_1^{-n}}\cos n(\theta-\phi)\right)\mathrm{d}\phi$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi} f_2(\phi)\left(\frac{\ln r/r_1}{\ln r_2/r_1} + 2\sum_{n=1}^{\infty}\frac{r^n - r_1^{2n}r^{-n}}{r_2^n - r_1^{2n}r_2^{-n}}\cos n(\theta-\phi)\right)\mathrm{d}\phi.$$

□

3.3 Half Upper-plane

In this section, Fourier Transform will be used to help solve boundary value problems on the upper half-plane for both Dirichlet and Neumann boundary conditions.

3.3.1 Dirichlet boundary condition

Consider the boundary value problem:

$$u_{xx} + u_{yy} = 0$$

$$u(x,0) = f(x)$$

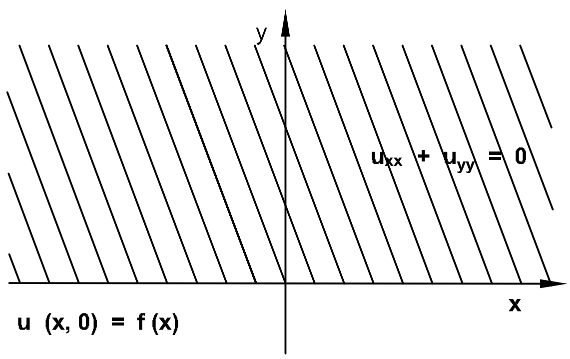

Figure 6: Dirichlet boundary condition on the upper half plane

The solution is

$$u(x,y) = \begin{cases} \dfrac{y}{\pi}\displaystyle\int_{-\infty}^{\infty} f(\tau)\frac{1}{y^2+(x-\tau)^2}\,\mathrm{d}\tau & y \neq 0, \\ f(x) & y = 0. \end{cases}$$

We make the following assumptions:

1. $f(x)$ satisfies the conditions for Fourier Transformation, that is, we can compute its corresponding $\hat{f}(\omega)$ by Fourier Transform formula[2].

2. $u(x,y)$ can be transformed with respect to x with Fourier Transform formula.

3. $\lim_{x\to\infty} u(x,y) = 0 = \lim_{x\to-\infty} u(x,y)$, and $\lim_{y\to\infty} u(x,y) = 0$.

First, we find the relation between the transformation of $u(x,y)$ and of $u_x(x,y)$,

$$
\begin{aligned}
\hat{u}(\omega,y) &= \int_{-\infty}^{\infty} u(x,y)e^{-i\omega x}\,\mathrm{d}x \\
&= \left.\frac{ue^{-i\omega x}}{-i\omega}\right|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} u_x \frac{e^{-i\omega x}}{-i\omega}\,\mathrm{d}x \\
&= \frac{1}{i\omega}\hat{u}_x(\omega,y).
\end{aligned}
$$

By induction it can be generalized to

$$\hat{u}^{(n)}(\omega,y) = (i\omega)^n \hat{u}(\omega,y).$$

Now, apply the Fourier transform formula to the Laplace's equation:

$$\mathscr{F}(u_{xx}+u_{yy}) = \mathscr{F}(0)$$

$$(i\omega)^2\hat{u}(\omega,y) + \hat{u}_{yy}(\omega,y) = 0 \qquad \text{(by property of linearity)}$$

Solving the second-order differential equation respecting y gives

$$\hat{u}(\omega,y) = C_1(\omega)e^{\omega y} + C_2(\omega)e^{-\omega y}.$$

Since the function is defined as y goes to ∞ for all $\omega \in \mathbb{R}$, $\hat{u}(\omega,y)$ should decay as y approaches ∞ . Therefore, it implies

$$\hat{u}(\omega,y) = C(\omega)e^{-|\omega|y}.$$

Computing the Fourier transform of the boundary condition $u(x,0) = f(x)$ gives

$$\hat{u}(\omega,0) = \hat{f}(\omega) = C(\omega),$$

which yields

$$\hat{u}(\omega,y) = \hat{f}(\omega)e^{-|\omega|y}.$$

Next, by inverse Fourier transform, we have

[2] See Appendix B.

$$
\begin{aligned}
u(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-|\omega|y}e^{i\omega x}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}\,\mathrm{d}\tau\right)e^{i\omega x-|\omega|y}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\int_{-\infty}^{\infty} e^{i\omega(x-\tau)-|\omega|y}\,\mathrm{d}\omega\,\mathrm{d}\tau \qquad \text{(by Fubini's theorem)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{0} e^{i\omega(x-\tau)-|\omega|y}\,\mathrm{d}\omega + \int_{0}^{\infty} e^{i\omega(x-\tau)-|\omega|y}\,\mathrm{d}\omega\right)\mathrm{d}\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{0} e^{i\omega(x-\tau)+\omega y}\,\mathrm{d}\omega + \int_{0}^{\infty} e^{i\omega(x-\tau)-\omega y}\,\mathrm{d}\omega\right)\mathrm{d}\tau.
\end{aligned}
$$

For $y > 0$, we have

$$
\begin{aligned}
u(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\left(\frac{1}{i(x-\tau)+y} + \frac{1}{-i(x-\tau)+y}\right)\mathrm{d}\tau \\
&= \frac{y}{\pi}\int_{-\infty}^{\infty} f(\tau)\frac{1}{y^2+(x-\tau)^2}\,\mathrm{d}\tau.
\end{aligned}
$$

For justification of solution when $y = 0$, it requires proper introduction of Fourier transform and theory of distributions, which we shall not discuss in detail here.

In conclusion,

$$
u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} f(\tau)\frac{1}{y^2+(x-\tau)^2}\,\mathrm{d}\tau, \qquad y > 0.
$$

□

3.3.2 Neumann boundary condition

Given the boundary value problem

$$
u_{xx} + u_{yy} = 0,
$$

$$
u_y(x,0) = g(x).
$$

The solution is

$$
u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\tau)\ln[y^2+(x-\tau)^2]\,\mathrm{d}\tau + C,
$$

where C is constant.

Almost everything from the Dirichlet problem follows the same, except that a different boundary condition is given. We again begin with the following assumptions:

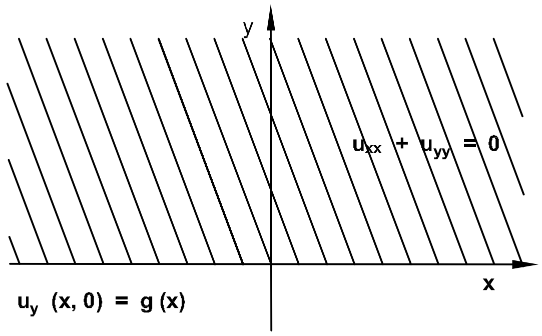

Figure 7: Neumann boundary condition on the upper half plane

1. $\lim_{x\to-\infty} u(x,y) = 0 = \lim_{x\to\infty} u(x,y)$, and $\lim_{y\to\infty} u(x,y) = 0$.
2. $g(x)$ has its Fourier Transform, that is, $\hat{g}(\omega)$ can be obtained by Fourier transform formula[3], and in particular, $\hat{g}(0) = \int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = 0$.
3. $$\frac{\partial}{\partial y}\mathscr{F}[u(x,y)] = \mathscr{F}[u_y(x,y)].$$

We already know from the last section the relation between the transform of $u(x,y)$ and the transform of $u_x(x,y)$ is

$$\begin{aligned}\hat{u}(\omega,y) &= \int_{-\infty}^{\infty} u(x,y)e^{-i\omega x}\,\mathrm{d}x \\ &= ue^{-i\omega x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_x \frac{e^{-i\omega x}}{-i\omega}\,\mathrm{d}x \\ &= \frac{1}{i\omega}\hat{u}_x(\omega,y),\end{aligned}$$

and by induction we have

$$\hat{u}^{(n)}(\omega,y) = (i\omega)^n\hat{u}(\omega,y).$$

On the basis of the above equation, we follow the Laplace's equation by Fourier transform:

$$u_{xx} + u_{yy} = 0,$$

$$\mathscr{F}(u_{xx} + u_{yy}) = 0,$$

$$-\omega^2\hat{u}(\omega,y) + \hat{u}_{yy}(\omega,y) = 0,$$

$$\hat{u}(\omega,y) = C_1(\omega)e^{\omega y} + C_2(\omega)e^{-\omega y}.$$

[3]See Appendix B.

As $y \to \infty$, $\hat{u}(\omega, y)$ should avoid infinity for any ω in $\mathbb{R}$. Therefore,

$$\hat{u}(\omega, y) = C(\omega)e^{-|\omega|y}. \tag{13}$$

In order to apply the boundary condition, we differentiate solution 13 with respect to y and transform the boundary condition $u_y(x,0) = g(x)$ respecting x, and obtain the following:

$$\hat{u}_y(\omega, y) = \begin{cases} -|\omega|C(\omega)e^{-|\omega|y} & \omega \neq 0 \\ 0 & \omega = 0 \end{cases},$$

$$\hat{u}_y(\omega, 0) = \hat{g}(\omega).$$

For the above equations to be continuous at $y = 0$, it requires

$$0 = \hat{g}(0),$$

which is satisfied by assumption 2.

Hence, solving for $C(\omega)$ in terms of $\hat{g}(\omega)$ gives

$$\hat{u}(\omega, y) = \begin{cases} -\frac{\hat{g}(\omega)}{|\omega|}e^{-|\omega|y} & \omega \neq 0 \\ C & \omega = 0 \end{cases}, \qquad \text{and}$$

$$\hat{u}_y(\omega, y) = \hat{g}(\omega)e^{-|\omega|y}. \tag{14}$$

Following equation 14, for $y > 0$,

$$\begin{aligned} u_y(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(\tau)e^{-i\omega\tau}\,\mathrm{d}\tau\right)e^{-|\omega|y}e^{i\omega x}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\tau)\int_{-\infty}^{\infty} e^{i\omega(x-\tau)-|\omega|y}\,\mathrm{d}\omega\,\mathrm{d}\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\tau)\left(\int_{-\infty}^{0} e^{\omega[y+i(x-\tau)]}\,\mathrm{d}\omega + \int_{0}^{\infty} e^{-\omega[y-i(x-\tau)]}\,\mathrm{d}\omega\right)\mathrm{d}\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\tau)\left(\frac{1}{y+i(x-\tau)} + \frac{1}{y-i(x-\tau)}\right)\mathrm{d}\tau \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} g(\tau)\frac{y}{y^2+(x-\tau)^2}\,\mathrm{d}\tau\,. \end{aligned}$$

For $y = 0$, we shall again avoid the formal justification of Fourier transform and theory of distributions.

Now, $u(x,y)$ can be obtained by integrating y for $u_y(x,y)$:

$$\begin{aligned}
u(x,y) &= \frac{1}{\pi}\int\int_{-\infty}^{\infty} g(\tau)\frac{y}{y^2+(x-\tau)^2}\,\mathrm{d}\tau\,\mathrm{d}y \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} g(\tau)\int \frac{y}{y^2+(x-\tau)^2}\,\mathrm{d}y\,\mathrm{d}\tau \text{ (by Fubini's theorem)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\tau)\{\ln[y^2+(x-\tau)^2]+C(x,\tau)\}\,\mathrm{d}\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} C(x,\tau)g(\tau)\,\mathrm{d}\tau + \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\tau)\ln[y^2+(x-\tau)^2]\,\mathrm{d}\tau. \qquad (15)
\end{aligned}$$

Similar to the Neumann problem on the circle, we can rewrite equation 15 in homogeneous and particular parts:

$$u(x,y) = u_h(x) + u_p(x,y),$$

where $u_h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(x,\tau)g(\tau)\,\mathrm{d}\tau$ and $u_p(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\tau)\ln[y^2+(x-\tau)^2]\,\mathrm{d}\tau$.

If we think of $u_p(x,y)$ as the particular solution to

$$u_{xx} + u_{yy} = 0,$$

$$u_y(x,0) = g(x),$$

$u_h(x)$ must adhere to the following

$$u_h''(x) = 0,$$

$$u_y(x,0) = 0.$$

This gives us $u_h(x) = Ax + B$. However, $u(x,y) \to 0$ as $x \to \pm\infty$ implies $u_h(x) \to 0$ as well. Therefore, for boundedness of $u_h(x,y)$, we conclude that $A = 0$. Then, $u_h(x) = C$, where C is constant.

Hence, the final solution is

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\tau)\ln[y^2+(x-\tau)^2]\,\mathrm{d}\tau + C.$$

□

3.4 Semi-infinite Stripe

In this case, we will use both the method of separation of variables and Fourier transform to find solution to the Laplace's equation,

$$u_{xx} + u_{yy} = 0,$$

$$u(0, y) = f_1(y),$$

$$u(x, 0) = f_2(x),$$

$$u(a, y) = f_3(y);$$

on semi-infinite stripe, as shown in Figure 7. If we separate the variables of $u(x, y)$ into $X(x)$ and $Y(y)$, we obtain $X''Y + XY'' = 0$. This implies $X'' = -\lambda X$ and $Y'' = \lambda Y$.

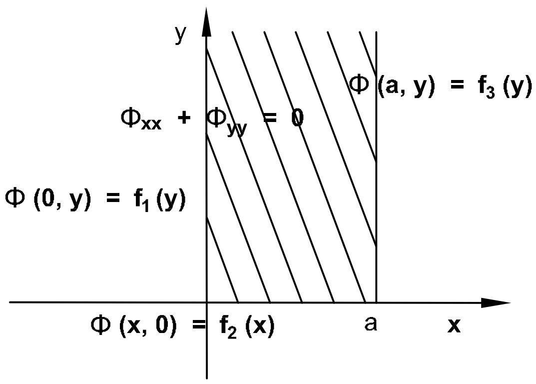

Figure 8: Half infinite stripe

Part 1 We first find a particular solution to

$$u_{xx} + u_{yy} = 0$$

$$u(0, y) = 0$$

$$u(a, y) = 0$$

$$u(x, 0) = f_2(x).$$

As we have dealt with this eigenvalue problems before, we know that

$$\begin{cases} X'' = -\lambda X \\ X(0) = 0 \\ X(a) = 0 \end{cases}$$

gives $\lambda = (\frac{n\pi}{a})^2$ and $X_n = \sin\left(\frac{n\pi}{a}x\right)$ with $n = 1, 2, 3...$ Then, $Y'' = \lambda Y = (\frac{n\pi}{a})^2 y$ gives

$$Y_n = C_1 e^{\frac{n\pi}{a}y} + C_2 e^{-\frac{n\pi}{a}y}.$$

For $Y(y)$ to be defined on $y \in (0,\infty)$, the above equation is reduced to $Y_n = C_n e^{-\frac{n\pi}{a}y}$. Combining $X_n(x)$ and $Y_n(y)$, we get

$$u(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi}{a}y} \sin\frac{n\pi}{a}x.$$

Apply the boundary condition $u(x,0) = f_2(x)$ to obtain

$$f_2(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{a}x.$$

By Fourier series formula, we have

$$C_n = \frac{2}{a}\int_0^a f_2(x)\sin\frac{n\pi}{a}x\,\mathrm{d}x.$$

Simplify our solution

$u(x,y)$

$$= \sum_{n=1}^{\infty}\left(\frac{2}{a}\int_0^a f_2(x)\sin\frac{n\pi}{a}x\,\mathrm{d}x\right)e^{-\frac{n\pi}{a}y}\sin\frac{n\pi}{a}x$$

$$= \frac{2}{a}\int_0^a f_2(\tau)\sum_{n=1}^{\infty} e^{-\frac{n\pi}{a}y}\sin\frac{n\pi}{a}\tau\sin\frac{n\pi}{a}x\,\mathrm{d}\tau$$

$$= \frac{1}{a}\int_0^a f_2(\tau)\sum_{n=1}^{\infty} e^{-\frac{n\pi}{a}y}\left(-\cos\frac{n\pi}{a}(\tau+x)+\cos\frac{n\pi}{a}(\tau-x)\right)\,\mathrm{d}\tau$$

$$= \frac{1}{a}\int_0^a f_2(\tau)Re\sum_{n=1}^{\infty}\left[\left(e^{\frac{\pi}{a}[-y+i(\tau-x)]}\right)^n - \left(e^{\frac{\pi}{a}[-y+i(\tau+x)]}\right)^n\right]\,\mathrm{d}\tau$$

$$= \frac{1}{a}\int_0^a f_2(\tau)Re\left[\frac{e^{\frac{\pi}{a}[-y+i(\tau-x)]}}{1-e^{\frac{\pi}{a}[-y+i(\tau-x)]}} - \frac{e^{\frac{\pi}{a}[-y+i(\tau+x)]}}{1-e^{\frac{\pi}{a}[-y+i(\tau+x)]}}\right]\,\mathrm{d}\tau$$

$$= \frac{2}{a}\int_0^a f_2(\tau)\frac{e^{-\frac{\pi}{a}y}(1-e^{-2\frac{\pi}{a}y})\sin\frac{\pi}{a}\tau\sin\frac{\pi}{a}x}{\left(1+e^{-2\frac{\pi}{a}y}-2e^{-\frac{\pi}{a}y}\cos\frac{\pi}{a}(\tau-x)\right)\left(1+e^{-2\frac{\pi}{a}y}-2e^{-\frac{\pi}{a}y}\cos\frac{\pi}{a}(\tau+x)\right)}\,\mathrm{d}\tau$$

(Write them in form of hyperbolic functions)

$$= \frac{2}{a}\int_0^a f_2(\tau)\frac{(\cosh\frac{\pi}{a}y-\sinh\frac{\pi}{a}y)[1-(\cosh\frac{\pi}{a}y)^2-(\sinh\frac{\pi}{a}y)^2+2\sinh\frac{\pi}{a}y\cosh\frac{\pi}{a}y]\sin\frac{\pi}{a}\tau\sin\frac{\pi}{a}x}{\begin{array}{c}\{1+(\cosh\frac{\pi}{a}y)^2+(\sinh\frac{\pi}{a}y)^2-2\sinh\frac{\pi}{a}y\cosh\frac{\pi}{a}y-2(\cosh\frac{\pi}{a}y-\sinh\frac{\pi}{a}y)(\cos\frac{\pi}{a}(\tau-x)\}\\ \{1+(\cosh\frac{\pi}{a}y)^2+(\sinh\frac{\pi}{a}y)^2-2\sinh\frac{\pi}{a}y\cosh\frac{\pi}{a}y-2(\cosh\frac{\pi}{a}y-\sinh\frac{\pi}{a}y)(\cos\frac{\pi}{a}(\tau+x)\}\end{array}}\,\mathrm{d}\tau$$

(Recall that$(\cosh\frac{\pi}{a}y)^2-(\sinh\frac{\pi}{a}y)^2 = 1$, factor the numerator and denominator by $(\cosh\frac{\pi}{a}y$ $-\sinh\frac{\pi}{a}y)^2$ and use the trigonometric identity $\cos\frac{\pi}{a}(\tau\pm x) = \cos\frac{\pi}{a}\tau\cos\frac{\pi}{a}x \mp \sin\frac{\pi}{a}\tau\sin\frac{\pi}{a}x$)

$$= \frac{1}{a}\int_0^a f_2(\tau)\frac{\sin\frac{\pi}{a}\tau\sin\frac{\pi}{a}x\sinh\frac{\pi}{a}y}{(\cosh\frac{\pi}{a}y-\cos\frac{\pi}{a}\tau\cos\frac{\pi}{a}x-\sin\frac{\pi}{a}\tau\sin\frac{\pi}{a}x)(\cosh\frac{\pi}{a}y-\cos\frac{\pi}{a}\tau\cos\frac{\pi}{a}x+\sin\frac{\pi}{a}\tau\sin\frac{\pi}{a}x)}\,\mathrm{d}\tau$$

$$= \frac{1}{a}\int_0^a f_2(\tau)\frac{\sin\frac{\pi}{a}\tau\sin\frac{\pi}{a}x\sinh\frac{\pi}{a}y}{(\cosh\frac{\pi}{a}y)^2+(\cos\frac{\pi}{a}\tau\cos\frac{\pi}{a}x)^2-2\cos\frac{\pi}{a}x\cos\frac{\pi}{a}\tau\cosh\frac{\pi}{a}y-(\sin\frac{\pi}{a}\tau)^2(\sin\frac{\pi}{a}x)^2}\,\mathrm{d}\tau$$

(Use the identity of difference of two squares)

$$= \frac{1}{a}\int_0^a f_2(\tau)\frac{\sin\frac{\pi}{a}\tau\sin\frac{\pi}{a}x\sinh\frac{\pi}{a}y}{(\cosh\frac{\pi}{a}y)^2+(\cos\frac{\pi}{a}\tau)^2(\cos\frac{\pi}{a}x)^2-2\cos\frac{\pi}{a}x\cos\frac{\pi}{a}\tau\cosh\frac{\pi}{a}y+((\cos\frac{\pi}{a}\tau)^2-1)(\sin\frac{\pi}{a}x)^2}\,\mathrm{d}\tau$$

(Use the trigonometric identity)

$$= \frac{1}{a}\int_0^a f_2(\tau)\frac{\sin\frac{\pi}{a}\tau\sin\frac{\pi}{a}x\sinh\frac{\pi}{a}y}{(\cosh\frac{\pi}{a}y\tau)^2-2\cos\frac{\pi}{a}x\cos\frac{\pi}{a}\tau\cosh\frac{\pi}{a}y+(\cos\frac{\pi}{a}x\cosh\frac{\pi}{a}y)^2-(\cos\frac{\pi}{a}x\cosh\frac{\pi}{a}y)^2+(\cosh\frac{\pi}{a}y)^2-(\sin\frac{\pi}{a}x)^2}\,\mathrm{d}\tau$$

$$= \frac{1}{a}\int_0^a f_2(\tau)\frac{\sin\frac{\pi}{a}\tau\sin\frac{\pi}{a}x\sinh\frac{\pi}{a}y}{(\cos\frac{\pi}{a}\tau-\cos\frac{\pi}{a}x\cosh\frac{\pi}{a}y)^2+(\sin\frac{\pi}{a}x)^2(\sinh\frac{\pi}{a}y)^2}\,\mathrm{d}\tau. \tag{16}$$

Part 2 Now we can consider a particular solution to

$$u_{xx}+u_{yy}=0, \tag{17}$$

$$u(x,0)=0, \tag{18}$$

$$u(0,y)=0, \tag{19}$$

$$u(a,y)=f_3(y). \tag{20}$$

Since $f_3(y)$ is only defined on $y>0$, in order to apply Fourier transform formula, we extend our boundary from $y\in(0,\infty)$ to $y\in(-\infty,\infty)$.

Recall that by assuming $\lim_{y\to\infty} u(x,y)=0$ we have $\hat{f}^{(n)}(\omega)=(i\omega)^n\hat{f}(\omega)$. If applying Fourier transform formula with respect to y on equation 17 $u_{xx}+u_{yy}=0$, we get

$$\hat{u}_{xx}(x,\omega)+(i\omega)^2\hat{u}(x,\omega)=0$$

$$\hat{u}_{xx}(x,\omega)=\omega^2\hat{u}(x,\omega)$$

$$\Longrightarrow \hat{u}(x,\omega)=C_1e^{\omega x}+C_2e^{-\omega x}. \tag{21}$$

With no justification, we claim that odd extension

$$F_3(y)=\begin{cases} f_3(y) & y>0 \\ f_3(-y) & y<0 \end{cases}$$

is sufficient to be comply with boundary conditions 18.

If applying Fourier transform formula to all of the equations 17, 18, 19 and 20, the boundary value problem is equivalent to

$$\hat{u}(x,\omega) = C_1 e^{\omega x} + C_2 e^{-\omega x}, \tag{22}$$

$$\int_{-\infty}^{\infty} \hat{u}(x,\omega)\,\mathrm{d}\omega = 0, \tag{23}$$

$$\hat{u}(0,\omega) = 0, \tag{24}$$

$$\hat{u}(a,\omega) = \hat{F}_3(\omega), \tag{25}$$

where $\hat{u}(x,\omega) = \mathscr{F}(u(x,y))$ and $\hat{F}_3(\omega) = \mathscr{F}(F(y)) = \int_0^{\infty} f_3(y)(e^{-i\omega y} - e^{i\omega y})\,\mathrm{d}y$.

Combining equations 22, 24 and 25, we find values of coefficients

$$C_1 = \frac{\hat{F}_3(\omega)}{e^{\omega a} - e^{-\omega a}}$$

$$C_2 = -\frac{\hat{F}_3(\omega)}{e^{\omega a} - e^{-\omega a}}.$$

Then, we check whether the solution

$$\hat{u}(x,\omega) = \hat{F}_3(\omega)\frac{e^{\omega x} - e^{-\omega x}}{e^{\omega a} - e^{-\omega a}} = \int_0^{\infty} f_3(\tau)(e^{-i\omega\tau} - e^{i\omega\tau})\frac{e^{\omega x} - e^{-\omega x}}{e^{\omega a} - e^{-\omega a}}\,\mathrm{d}\tau$$

satisfies condition 23.

It is expected

$$\int_{-\infty}^{\infty} u(x,\omega)\,\mathrm{d}\omega = \int_0^{\infty} f_3(\tau)\int_{-\infty}^{\infty} (e^{-i\omega\tau} - e^{i\omega\tau})\frac{e^{\omega x} - e^{-\omega x}}{e^{\omega a} - e^{-\omega a}}\,\mathrm{d}\omega\,\mathrm{d}\tau = 0.$$

It then follows

$$\begin{aligned} u(x,y) &= \int_{-\infty}^{\infty}\left(\int_0^{\infty} F_3(\tau)(e^{-i\omega\tau} - e^{i\omega\tau})\,\mathrm{d}\tau\right)\left(\frac{e^{\omega x} - e^{-\omega x}}{e^{\omega a} - e^{-\omega a}}\right)e^{i\omega y}\,\mathrm{d}\omega \\ &= \int_0^{\infty} f_3(\tau)\int_0^{\infty} \frac{(e^{-i\omega\tau} - e^{i\omega\tau})(e^{i\omega y} - e^{-i\omega y})(e^{\omega x} - e^{-\omega x})}{e^{\omega a} - e^{-\omega a}}\,\mathrm{d}\omega\,\mathrm{d}\tau \\ &= 4\int_0^{\infty} f_3(\tau)\int_0^{\infty} \frac{\sin\omega\tau\,\sin\omega y\,\sinh\omega x}{\sinh\omega a}\,\mathrm{d}\omega\,\mathrm{d}\tau. \end{aligned}$$

Part 3 We now repeat the steps in Part 2. Let us consider the following boundary value problem

$$u_{xx} + u_{yy} = 0,$$

$$u(x,0) = f_1(y),$$

$$u(0,y) = 0,$$

$$u(a,y) = 0.$$

Recall that by Fourier transform under the same assumptions we can obtain

$$\hat{u}(x,\omega) = C_1 e^{\omega x} + C_2 e^{-\omega x}, \tag{26}$$

$$\int_{-\infty}^{\infty} \hat{u}(x,\omega)\,\mathrm{d}\omega = 0, \tag{27}$$

$$\hat{u}(0,\omega) = \hat{F}_1(\omega), \tag{28}$$

$$\hat{u}(a,\omega) = 0,, \tag{29}$$

where $\hat{F}_1(\omega) = \int_0^{\infty} f_1(y)(e^{-i\omega y} - e^{i\omega y})\,\mathrm{d}y = -2i\int_0^{\infty} f_1(y)\sin\omega y\,\mathrm{d}y.$

By solving equation 26 with boundary conditions 28 and 29 gives us

$$\begin{aligned}
\hat{u}(x,\omega) &= \hat{F}_1(\omega)\frac{e^{\omega x} - e^{\omega(2a-x)}}{1-e^{2a\omega}} \\
&= \hat{F}_1(\omega)\frac{\sinh\omega(a-x)}{\sinh\omega a} \\
&= -2i\int_0^{\infty} f_1(\tau)\frac{\sin\omega\tau\sinh\omega(a-x)}{\sinh\omega a}\,\mathrm{d}\tau.
\end{aligned}$$

By the inverse Fourier transform formula, we have

$$\begin{aligned}
u(x,y) &= \int_{-\infty}^{\infty} \hat{u}(x,\omega)e^{i\omega y}\,\mathrm{d}\omega \\
&= (-2i)\int_0^{\infty} f_1(\tau)\int_0^{\infty} \frac{\sin\omega\tau\,\sinh\omega(a-x)}{\sinh\omega a}(e^{i\omega y} - e^{-i\omega y})\,\mathrm{d}\omega\,\mathrm{d}\tau \\
&= 4\int_0^{\infty} f_1(\tau)\int_0^{\infty} \frac{\sin\omega\tau\,\sin\omega y\,\sinh\omega(a-x)}{\sinh\omega a}\,\mathrm{d}\omega\,\mathrm{d}\tau
\end{aligned}$$

Put together three paritcular solutions, we have

$$\begin{aligned}
u(x,y) &= 4\int_0^{\infty} f_1(\tau)\int_0^{\infty} \frac{\sin\omega\tau\,\sin\omega y\,\sinh\omega(a-x)}{\sinh\omega a}\,\mathrm{d}\omega\,\mathrm{d}\tau \\
&+ \frac{1}{a}\int_0^{a} f_2(\tau)\frac{\sin\frac{\pi}{a}\tau\sin\frac{\pi}{a}x\sinh\frac{\pi}{a}y}{(\cos\frac{\pi}{a}\tau - \cos\frac{\pi}{a}x\cosh\frac{\pi}{a}y)^2 + (\sin\frac{\pi}{a}x)^2(\sinh\frac{\pi}{a}y)^2}\,\mathrm{d}\tau
\end{aligned}$$

$$+4\int_0^\infty f_3(\tau)\int_0^\infty \frac{\sin\omega\tau\,\sin\omega y\,\sinh\omega x}{\sinh\omega a}\,\mathrm{d}\omega\,\mathrm{d}\tau.$$

□

4 Conformal Mapping

This section introduces to conformal mapping. The first few theorems provide intuition for construction of conformal mapping. Though we are not expected to construct any mapping on our own, we shall briefly look through the theorems, as they will help us understand the conformality of two particular mapping methods, namely Möbius Transformation and Schwarz-Christoffel Transformation later in this section. Our ultimate goal of conformal mapping is to solve Laplace's equations on complex domains by reducing to Laplace's equations on simplified domains using conformal mapping.

4.1 Introduction to Conformal Mapping

Theorem 2 (Riemann Mapping Theorem)**.** *If D is simply connected and not the entire plane, then there is a one-to-one analytic function that maps D onto the open unit disk.*

Interested readers can find the proof in the book Real and Complex Analysis[4]. We will skip it here.

Lemma 1 (Inverse function theorem)**.** *If f is analytic at z_0 and $f'(z_0)\neq 0$, then there is an open disk D centered at z_0 such that f is one-to-one on D.* [5]

Remark The Lemma helps to find a local one-to-one mapping around a certain point. However, in order to find a one-to-one function on a particular domain, the function has to be analytic and has non-vanishing first derivative at every point in the domain. We can select an arbitrary point in the domain and prove that the function is one-to-one at a neighborhood of the point.

[4] Reference [2]

Definition. *A map $f(z)$ is said to be conformal at z_0 if it is analytic with nonvanishing first-order derivative $f'(z_0)$ at z_0.*

Lemma 2 (Rotation matrix). *In two dimension, a rotation matrix $R(\theta)$ that rotates points in xy-plane counterclockwise through an angle θ about the origin is given by*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Lemma 3 (Jacobian matrix). *Jacobian matrix is the matrix of all first-order partial derivatives of a vector-valued function. For $f:(x,y)\to(u,v)$, the Jacobian matrix is*

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Theorem 3 (Local conformality). *If a function f is analytic at point z_0 with $f'(z_0)\neq 0$, then any nonzero acute angle between any two directed smooth curves intersecting at z_0 is preserved under the mapping f.*

Proof. Recall that the Taylor series extension for $f(z)$ at z_0 is

$$f(z) = f(z_0) + \mathbf{J}(f(z_0))\delta z + \epsilon(\delta z^2),$$

where $\mathbf{J}(f(z_0))$ is a jacobian matrix.
When δz is small, the high order terms are negligible, and it is left with

$$f(z) \approx f(z_0) + \mathbf{J}(f(z_0))\delta z.$$

We will use the rotation matrix and Jacobian matrix to justify angle preservation of conformal mappings. Suppose an analytic function $f(z)$ with $f'(z_0)\neq 0$ maps from z plane into w plane, where $z = x+iy$ and $w = u+iv$. The Jacobian matrix of f on disk $D(z_0,r)$ for some $r>0$ such that $f'(z)\neq 0$ for any $z\in D$ is

$$J[f(z)] = \begin{pmatrix} u_x & u_v \\ v_x & v_y \end{pmatrix}$$

$$= \begin{pmatrix} u_x & u_v \\ -u_y & u_x \end{pmatrix} \qquad \text{(by Cauchy-Riemann equations)}$$

$$= \sqrt{u_x^2+u_y^2}\begin{pmatrix} \dfrac{u_x}{\sqrt{u_x^2+u_y^2}} & \dfrac{u_y}{\sqrt{u_x^2+u_y^2}} \\ \dfrac{-u_y}{\sqrt{u_x^2+u_y^2}} & \dfrac{u_x}{\sqrt{u_x^2+u_y^2}} \end{pmatrix}.$$

Since $f'(z_0) \neq 0$, then by continuity of f', $f'(z) \neq 0$ on D for some $r > 0$. This implies $u_x + iv_x = u_x + iu_y \neq 0$, which implies $u_x \neq 0$ and $u_y \neq 0$ on the disk D.
Thus, it yields

$$J[f(z)] = \sqrt{u_x^2+u_y^2}\, R(-\theta),$$

where $\cos\theta = \dfrac{u_x}{\sqrt{u_x^2+u_y^2}}$, $\sin\theta = \dfrac{u_y}{\sqrt{u_x^2+u_y^2}}$,
and $R(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is a rotation matrix clockwise around z_0.

We assume r is sufficiently small so that on the disk D

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \mathbf{J}(f(z_0))\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

holds.

Therefore, the mapping $f(z)$ rotates every point in some neighborhood around z_0 through the fixed angle θ clockwise about z_0. Furthermore, the angle between any two curves intersecting at z_0 are preserved. □

Theorem 4 (Global conformality)**.** *If $f(z)$ is conformal at every point on domain D, then it is said to be conformal on the domain.*

4.2 Möbius Transformation

Definition. *A Möbius transformation is complex-valued function in form of*

$$w = f(z) = \frac{az+b}{cz+d}$$

where a, b, c and d are complex constants such that $ad \neq bc$.

We shall begin checking its conformality:

- analyticity: $f(z)$ is analytic everywhere except at $-\dfrac{d}{c}$.

- one-to-oneness: it is one-to-one on its domain because $f'(z) = \dfrac{ad-bc}{(cz+d)^2} \neq 0$.

Therefore, $f(z)$ is conformal on $\mathbb{C} - \{-\dfrac{d}{c}\}$.

Below are some properties of Möbius transformation.

1. Linear transformation: It is a combination of translation, scaling, rotation, and inversion. Symbolically, $w = z+b$, $w = az$ where a is real, $w = e^{i\theta}z$ and $w = \frac{1}{z}$.

2. Extended domain: It maps from $C\cup\{\infty\}$ to $C\cup\{\infty\}$, where ∞ is treated as one point regardless of its location.

3. Mapping of circles and lines: Class of circles and lines are mapped to themselves. In the table below is the conclusion on mappings between lines and circles.[6]

Curve type	Through origin	Inversion curve	Through origin
Line	Yes	Line	Yes
Line	No	Circle	Yes
Circle	Yes	Line	No
Circle	No	Circle	No

4. Inverse mapping The inverse of Möbius transform is also Möbius transform. Given $w = f(z) = \dfrac{az+b}{cz+d}$, where $ad \neq bc$, its inverse function is $z = f^{-1}(w) = \dfrac{dw-b}{a-cw}$, which is also the form of Möbius transform. Since $f(z)$ is one-to-one, the inverse function $f^{-1}(z)$ is unique.

5. Composition of mappings The composition of Möbius transform is also Möbius transform. Let $w_2 = \dfrac{aw_1+b}{cw_2+d}$ and $w_3 = \dfrac{ew_2+f}{gw_2+h}$ such that $ad \neq bc$ and $eh \neq fg$. We want to show that the composite function $w_3(w_2(w_1))$ is also Möbius transform. The composite function

$$
\begin{aligned}
w_3(w_2(w_1)) &= \frac{ew_2+f}{gw_2+h} \\
&= \frac{e(aw_1+b)+f(cw_1+d)}{g(aw_1+b)+h(cw_1+d)} \\
&= \frac{(ea+fc)w_1+(eb+fd)}{(ga+hc)w_1+(gb+hd)}
\end{aligned}
$$

[6] Proof given in Appendix C.

is Möbius transform because of

$$(ea+fc)(gb+hd)-(eb+fd)(ga+hc)=(ad-bc)(eh-fg)\neq 0.$$

Besides, there are some techniques that help construct Möbius Transformation.

Theorem 5 (Cross-ratio Method)**.** *For any three points z_1, z_2 and z_3, the mapping to* 0*,* 1 *and* ∞ *is*

$$T(z)=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

Remark In general, for any other three points w_1,w_2 and w_3, we can apply the Cross-ratio method

$$T(w)=\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

again. Then, we can find the Möbius mapping from three arbitrary points z_1, z_2 and z_3 to another three points w_1, w_2 and w_3,

$$w=\frac{w_1(z_2-z_1)(w_2-w_3)(z-z_3)-w_3(z_2-z_3)(w_2-w_1)(z-z_1)}{(z_2-z_1)(w_2-w_3)(z-z_3)-(z_2-z_3)(w_2-w_1)(z-z_1)}.$$

A mapping between lines and circles has a flexibility of three points. That is why cross-ratio method is effective in mapping between circles and lines.

Definition (Symmetry)**.** *Two points z_1 and z_2 are said to be symmetric with respect to a circle C if every straight line or circle passing through z_1 and z_2 intersects C orthogonally.*

Theorem 6 (Symmetry Principle)**.** *Let $w=f(z)$ be a Möbius transform, and circle C_w be the image of circle C_z, ie, $C_w=f(C_z)$. Two points w_1 and w_2 are symmetric with respect to circle C_w under the Möbius transform $w_1=f(z_1)$ and $w_2=f(z_2)$ if and only if z_1 and z_2 are symmetric with respect to circle C_z.*

Remark Symmetry principle simplifies the cross-ratio method in many cases, especially in the case of mapping any circles to circles with certain center.

A good example is applying symmetry principle to map two separate circles to two concentric circles at the origin, so that the exterior of two original circles will be mapped onto an annulus centered at the origin. The main idea is to find two points symmetric to both circles and map one of them to 0 and the other to ∞, then the choice of mapping a third point can be used to decide the radius of the annulus.

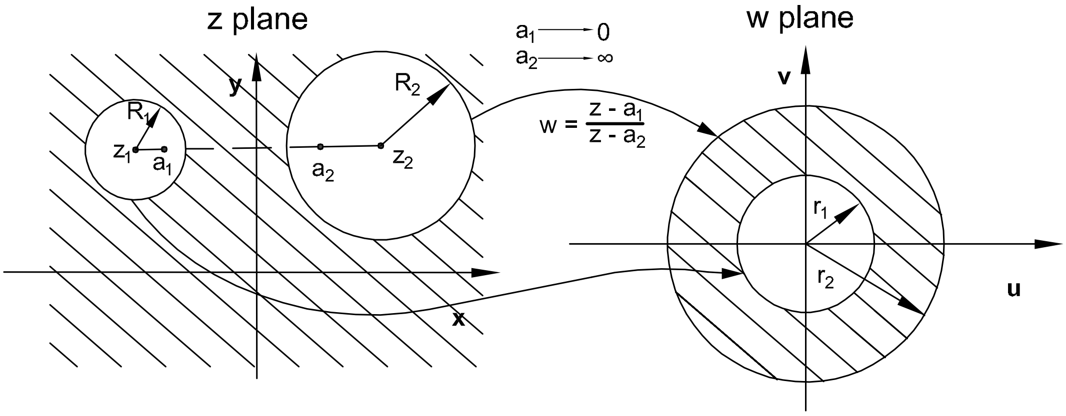

Figure 9: Mapping of symmetric points

First, by symmetric formula, we have

$$\begin{cases} a_2 = z_1 + \dfrac{R_1^2}{\overline{a}_1 - \overline{z}_1} \\ a_2 = z_2 + \dfrac{R_2^2}{\overline{a}_1 - \overline{z}_2} \end{cases} \Longrightarrow \begin{cases} a_1 = \dfrac{(\overline{z}_1-\overline{z}_2)(z_1+z_2)+R_2^2-R_1^2+\sqrt{(R_2^2-R_1^2)^2+|z_1-z_2|^2\left[|z_1-z_2|^2-2(R_1^2+R_2^2)\right]}}{2(\overline{z}_1-\overline{z}_2)} \\ a_2 = \dfrac{(\overline{z}_1-\overline{z}_2)(z_1+z_2)+R_2^2-R_1^2-\sqrt{(R_2^2-R_1^2)^2+|z_1-z_2|^2\left[|z_1-z_2|^2-2(R_1^2+R_2^2)\right]}}{2(\overline{z}_1-\overline{z}_2)} \end{cases}.$$

Then, the Möbius transform that maps the two separate circles to the concentric ones is given by

$$w = C\frac{z-a_1}{z-a_2},$$

where c is a complex constant.

4.2.1 Application in solving Laplace's equations on separate circles

We are to find a solution $\phi(x,y)$ to the Laplace's equation $\phi_{xx}+\phi_{yy}=0$ on the z-domain, supposing the boundary conditions on z-domain are given by

$$f_1(z_1+R_1e^{i\alpha})$$

on circle $C_1(z_1,R_1)$ and

$$f_2(z_2+R_2e^{i\alpha})$$

on circle $C_2(z_2,R_2)$.

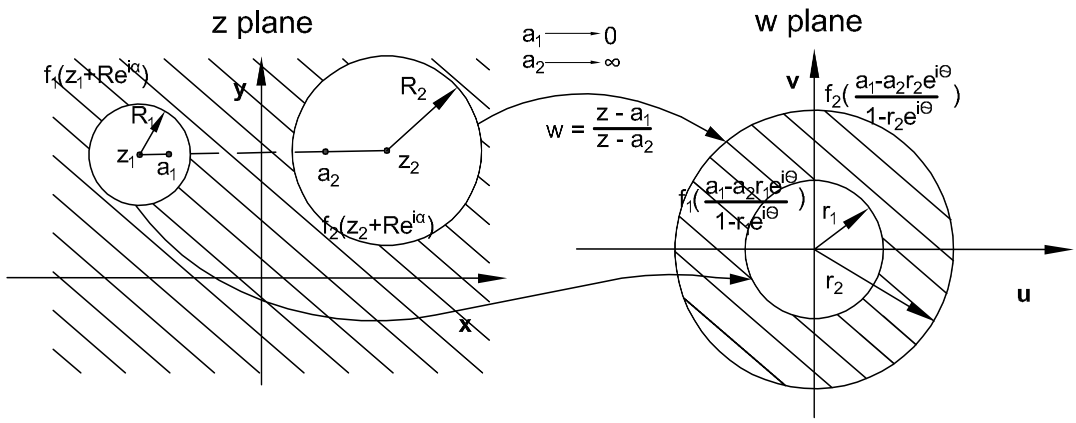

Figure 10: Solving Laplace's equation by mapping symmetric points

By properties of conformal mapping, we are able to first find a solution $\psi(r,\theta)$ to the Laplace's equation $r^2\psi_{rr} + r\psi_r + \psi_{\theta\theta} = 0$ on the w-domain and then find the solution to the original domain by changing the variables.

For simplicity, we set $C = 1$ in the mapping function, which yields

$$w = \frac{z-a_1}{z-a_2}, \tag{30}$$

where

$$\begin{cases} a_1 = \dfrac{(\bar{z}_1-\bar{z}_2)(z_1+z_2)+R_2^2-R_1^2+\sqrt{(R_2^2-R_1^2)^2+|z_1-z_2|^2\left[|z_1-z_2|^2-2(R_1^2+R_2^2)\right]}}{2(\bar{z}_1-\bar{z}_2)} \\ a_2 = \dfrac{(\bar{z}_1-\bar{z}_2)(z_1+z_2)+R_2^2-R_1^2-\sqrt{(R_2^2-R_1^2)^2+|z_1-z_2|^2\left[|z_1-z_2|^2-2(R_1^2+R_2^2)\right]}}{2(\bar{z}_1-\bar{z}_2)} \end{cases}.$$

We find radius of two circles $C(0,r_1)$ and $C(0,r_2)$ on the w-domain by the mapping function 30

$$r_1 = \left|\frac{z_1+R_1-a_1}{z_1+R_1-a_2}\right|$$

$$\text{and} \quad r_2 = \left|\frac{z_2+R_2-a_1}{z_2+R_2-a_2}\right|.$$

We need to find the corresponding boundary conditions on w-domain.
By letting $z = z_1 + R_1e^{i\alpha}$ and $w = r_1e^{i\theta}$, we have

$$w = \frac{z-a_1}{z-a_2}$$

$$\Longrightarrow \quad r_1e^{i\theta} = \frac{z_1+R_1e^{i\alpha}-a_1}{z_1+R_1e^{i\alpha}-a_2}$$

$$\implies \quad z_1 + R_1 e^{i\alpha} = \frac{a_1 - a_2 r_1 e^{i\theta}}{1 - r_1 e^{i\theta}}.$$

Similarly, by letting $z = z_2 + R_2 e^{i\alpha}$ and $w = r_2 e^{i\theta}$, we have

$$z_2 + R_2 e^{i\alpha} = \frac{a_1 - a_2 r_2 e^{i\theta}}{1 - r_2 e^{i\theta}}.$$

Therefore, the boundary conditions on circles $C(0, r_1)$ and $C(0, r_2)$ on w-domain are

$$f_1(z_1 + R_1 e^{i\alpha}) = f_1(\frac{a_1 - a_2 r_1 e^{i\theta}}{1 - r_1 e^{i\theta}})$$

and

$$f_2(z_2 + R_2 e^{i\alpha}) = f_2(\frac{a_1 - a_2 r_2 e^{i\theta}}{1 - r_2 e^{i\theta}}),$$

respectively.

Now, we just need to solve the Laplace's equation on the w-domain,

$$\begin{cases} r^2 \psi_{rr} + r\psi_r + \psi_{\theta\theta} = 0, \\ \psi(r_1, \theta) = f_1(\frac{a_1 - a_2 r_1 e^{i\theta}}{1 - r_1 e^{i\theta}}), \\ \psi(r_2, \theta) = f_2(\frac{a_1 - a_2 r_2 e^{i\theta}}{1 - r_2 e^{i\theta}}), \\ \psi(r, -\pi) = \psi(r, \pi), \\ \psi'(r, -\pi) = \psi'(r, \pi). \end{cases}$$

By the formula we derived in Section 4.2, we find the solution

$$\psi(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(\frac{a_1 - a_2 r_1 e^{i\theta}}{1 - r_1 e^{i\theta}})\left(\frac{\ln r/r_2}{\ln r_1/r_2} + 2\sum_{n=1}^{\infty}\frac{r^n - r_2^{2n} r^{-n}}{r_1^n - r_2^{2n} r_1^{-n}}\cos n(\theta - \phi)\right) \mathrm{d}\phi$$
$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} f_2(\frac{a_1 - a_2 r_2 e^{i\theta}}{1 - r_2 e^{i\theta}})\left(\frac{\ln r/r_1}{\ln r_2/r_1} + 2\sum_{n=1}^{\infty}\frac{r^n - r_1^{2n} r^{-n}}{r_2^n - r_1^{2n} r_2^{-n}}\cos n(\theta - \phi)\right) \mathrm{d}\phi. \tag{31}$$

We know that

$$r = |w| = \left|\frac{z - a_1}{z - a_2}\right| = \left|\frac{x + iy - a_1}{x + iy - a_2}\right|$$

$$\text{and} \quad \theta = \arg(w) = \arg\left(\frac{z - a_1}{z - a_2}\right) = \arg(\frac{x + iy - a_1}{x + iy - a_2}).$$

Let $t = \dfrac{a_1 - a_2 r_1 e^{i\phi}}{1 - r_1 e^{i\phi}}$, then $\phi = -i \log \dfrac{a_1 - t}{a_2 r_1 - r_1 t}$ and $\mathrm{d}\phi = \dfrac{i(a_2 - a_1)}{(a_1 - t)(a_2 - t)}\,\mathrm{d}t.$

Hence, we can can find an explicit form for $\phi(x, y)$ from equation 31

$$\phi(z(x,y)) = \frac{1}{2\pi}\int_{\frac{a_1 + a_2 r_1}{1 + r_1}}^{\frac{a_1 - a_2 r_1}{1 - r_1}} \frac{i(a_2 - a_1)}{(a_1 - t)(a_2 - t)} f_1(t)\Big(\frac{\ln|\frac{z - a_1}{z - a_2}| - \ln r_2}{\ln r_1 - \ln r_2}$$
$$+ 2\sum_{n=1}^{\infty}\frac{|\frac{z - a_1}{z - a_2}|^n - r_1^{2n}|\frac{z - a_2}{z - a_1}|^n}{r_1^n - r_2^{2n} r_1^{-n}}\cos n(\arg\frac{z - a_1}{z - a_2} + \frac{i(a_1 - t)}{a_2 r_1 - r_1 t}\Big)\,\mathrm{d}t$$

$$+\frac{1}{2\pi}\int_{\frac{a_1+a_2r_2}{1+r_2}}^{\frac{a_1-a_2r_2}{1-r_2}}\frac{i(a_2-a_1)}{(a_1-t)(a_2-t)}f_2(t)\Big(\frac{\ln|\frac{z-a_1}{z-a_2}|-\ln r_1}{\ln r_2-\ln r_1}$$

$$+2\sum_{n=1}^{\infty}\frac{|\frac{z-a_1}{z-a_2}|^n-r_2^{2n}|\frac{z-a_2}{z-a_1}|^n}{r_2^n-r_1^{2n}r_2^{-n}}\cos n(\arg\frac{z-a_1}{z-a_2}+\frac{i(a_1-t)}{a_2r_2-r_2t}\Big)\,\mathrm{d}t.$$

□

4.2.2 Solutions on a disk by mapping from upper-half plane

The Laplace's equation in polar form on $w-$ plane is

$$r^2\Phi_{rr}+r\Phi_r+\Phi_{\theta\theta}=0$$

$$\Phi(r,-\pi)=\Phi(r.\pi)$$

$$\Phi(\rho,\theta)=f(\theta)$$

Previously, we applied the method of separating variables to derive the solution to Laplace's

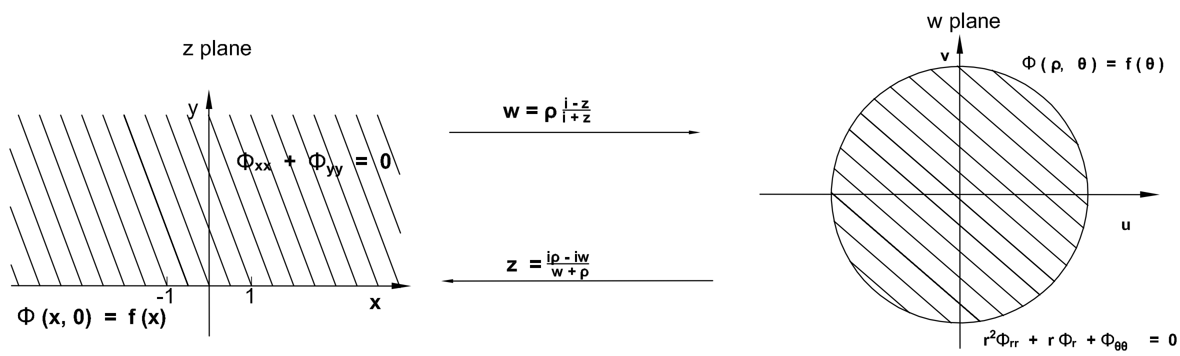

Figure 11: Upper-half plane to disk

equation on a disk with radius ρ. In this section, we will derive the solution by Möbius transform, and verify it with the solution we already knew.

The function that maps from the $z-$plane to the $w-$plane is

$$w=\rho\frac{i-z}{i+z}$$

and its inverse is

$$z=\frac{i\rho-iw}{w+\rho}.$$

If we write the inverse functions in the Cartesian form and take $u(r,\theta)=r\cos\theta$ and $v(r,\theta)=r\sin\theta$, we get

$$x+iy=\frac{2\rho v}{(u+\rho)^2+v^2}+i\frac{\rho^2-u^2-v^2}{(u+\rho)^2+v^2}=\frac{2\rho r\sin\theta}{\rho^2+r^2+2\rho r\cos\theta}+i\frac{\rho^2-r^2}{\rho^2+r^2+2\rho r\cos\theta}.$$

More specifically, at the boundary, two domains are related by $x = \dfrac{\sin\theta}{1+\cos\theta}$.

To avoid ambiguity, let $\tau = x$, $\phi = \theta$ and $g(\phi) = \dfrac{\sin\phi}{1+\cos\phi}$, then it follows that $\tau = g(\phi)$ and $\phi = g^{-1}(\tau)$. Furthermore, we notice that $\phi \to (-\pi)^+$ as $\tau \to -\infty$ and $\phi \to (\pi)^-$ as $\tau \to \infty$.

Applying the formula for Laplace's equation on upper-half plane, we obtain

$$\Phi(r,\theta) = \frac{y}{\pi}\int_{-\pi}^{\pi} \frac{f(g^{-1}(\tau))}{(y^2+(x-\tau)^2}\,\mathrm{d}\tau.$$

Since we have $g^{-1}(\tau) = \phi$ and $\tau = \dfrac{\sin\phi}{1+\cos\phi}$, we derive $\mathrm{d}\tau = \dfrac{1}{1+\cos\phi}\,\mathrm{d}\phi$, which yields

$$\begin{aligned}
&\Phi(r,\theta)\\
&= \frac{\rho^2-r^2}{\pi(\rho^2+r^2+2\rho r\cos\theta)}\int_{-\pi}^{\pi} \frac{f(\phi)}{\left[\left(\dfrac{\rho^2-r^2}{\rho^2+r^2+2\rho r\cos\theta}\right)^2+\left(\dfrac{2\rho r\sin\theta}{\rho^2+r^2+2\rho r\cos\theta}-\dfrac{\sin\phi}{1+\cos\phi}\right)^2\right](1+\cos\phi)}\,\mathrm{d}\phi\\
&= \frac{(\rho^2-r^2)(\rho^2+r^2+2\rho r\cos\theta)}{\pi}\int_{-\pi}^{\pi} \frac{f(\phi)}{\left[(\rho^2-r^2)^2+(2\rho r\sin\theta-\dfrac{\sin\phi}{1+\cos\phi}(\rho^2+r^2+2\rho r\cos\theta)\right]^2(1+\cos\phi)}\,\mathrm{d}\phi\\
&= \frac{(\rho^2-r^2)(\rho^2+r^2+2\rho r\cos\theta)}{\pi}\int_{-\pi}^{\pi} \frac{f(\phi)}{\begin{array}{c}(1+\cos\phi)[(\rho^2-r^2)^2+4\rho^2r^2\sin^2\theta]-4\rho r\sin\theta\sin\phi\\ (\rho^2+r^2+2\rho r\cos\theta)+(1-\cos\phi)(\rho^2+r^2+2\rho r\cos\theta)^2\end{array}}\,\mathrm{d}\phi\\
&= \frac{(\rho^2-r^2)(\rho^2+r^2+2\rho r\cos\theta)}{\pi}\int_{-\pi}^{\pi} \frac{f(\phi)}{\begin{array}{c}(1+\cos\phi)(\rho^2+r^2+2\rho r\cos\theta)(\rho^2+r^2-2\rho r\cos\theta)-4\rho r\\ \sin\theta\sin\phi(\rho^2+r^2+2\rho r\cos\theta)+(1-\cos\phi)(\rho^2+r^2+2\rho r\cos\theta)^2\end{array}}\,\mathrm{d}\phi\\
&= \frac{\rho^2-r^2}{\pi}\int_{-\pi}^{\pi} \frac{f(\phi)}{(1+\cos\phi)(\rho^2+r^2-2\rho r\cos\theta)-4\rho r\sin\theta\sin\phi+(1-\cos\phi)(\rho^2+r^2+2\rho r\cos\theta)}\,\mathrm{d}\phi\\
&= \frac{\rho^2-r^2}{\pi}\int_{-\pi}^{\pi} \frac{f(\phi)}{2\rho^2+2r^2-4\rho r\cos\theta\cos\phi-4\rho r\sin\theta\sin\phi}\,\mathrm{d}\phi\\
&= \frac{\rho^2-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{f(\phi)}{\rho^2+r^2-2\rho r\cos(\theta-\phi)}\,\mathrm{d}\phi.
\end{aligned}$$

□

The solution we just derived matches the one from section 3.1.1, therefore it is verified.

4.3 Schwarz-Christoffel Transformation

The Schwarz-Christoffel transform maps line segments on complex domains onto the real axis. We are interested in applying S.C. transformation to map polygonal domains onto upper-half plane so that we can solve the Laplace's equations on the simplified domain.

Theorem 7 (Schwarz-Christoffel Mapping Theorem)**.** *Let P be a positively oriented polygon with vertices w_1, w_2, ..., w_n with corresponding right-turn angles θ_1, θ_2, ..., θ_n. Then, a one-to-one conformal function that maps the upper-half plane onto the polygon is*

$$f(z) = A\int_0^z (\zeta - x_1)^{\frac{\theta_1}{\pi}}(\zeta - x_2)^{\frac{\theta_2}{\pi}}...(\zeta - x_{n-1})^{\frac{\theta_{n-1}}{\pi}}\,\mathrm{d}\zeta + B,$$

where $f(x_1) = w_1$, $f(x_2) = w_2$, ..., $f(x_{n-1}) = w_{n-1}$, $f(\infty) = w_n$, and A, B are complex constants.

4.3.1 A new look on Laplace's equation on semi-infinite stripe

Before, we found a solution to Laplace's equation on the semi-infinite stripe domain by the method of Fourier transform and separating variables. In this section, we will apply Schwarz-Christoffel transform and solution to Laplace's equation on upper half plane, which we already knew.

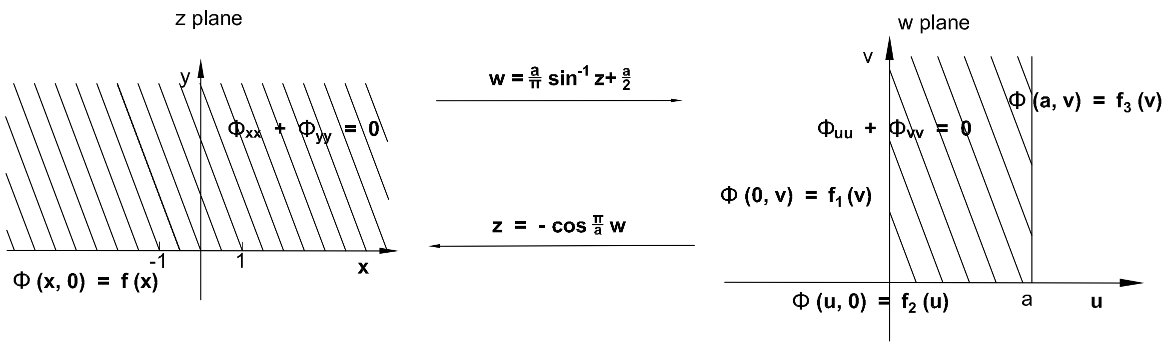

Figure 12: Mapping from the upper half plane to the semi-infinite stripe

By Theorem 6, choosing $x_1 = -1$ and $x_2 = 1$, the conformal function can be written as

$$w = C\int_0^z (\zeta^2 - 1)^{-\frac{1}{2}}\,\mathrm{d}\zeta$$

$$= A\sin^{-1} z + B, \tag{32}$$

where A and B are complex constants and $C = iA$.

Applying $0 = A\sin^{-1}(-1) + B$ and $a = A\sin^{-1}(1) + B$ to equation 32, we find the values of the coefficients

$$\begin{cases} A = \frac{a}{\pi}, \\ B = \frac{a}{2}. \end{cases}$$

Hence, the mapping function is

$$w = \frac{a}{\pi}\sin^{-1} z + \frac{a}{2}.$$

The inverse mapping function follows

$$z = \sin\frac{\pi}{a}(w - \frac{a}{2}) = -\sin(\frac{\pi}{2} - \frac{\pi}{a}w)$$

$$= -\cos(\frac{\pi}{a}w).$$

It will be useful to reduce the inverse mapping function and to Cartesian form and find the relation of $x(u,v)$ and $y(u.v)$

$$x + iy = -\cos\frac{\pi}{a}u\cosh\frac{\pi}{a}v + i\sin\frac{\pi}{a}u\sinh\frac{\pi}{a}v,$$

$$x = -\cos\frac{\pi}{a}u\cosh\frac{\pi}{a}v, \quad (33)$$

$$y = \sin\frac{\pi}{a}u\sinh\frac{\pi}{a}v. \quad (34)$$

In particular, when $w = iv$ $(v > 0)$, we have

$$x = -\cosh\frac{\pi}{a}v \Longleftrightarrow v = \frac{a}{\pi}\cosh^{-1}(-x);$$

when $w = a + iv$ $(v > 0)$, we have

$$x = \cosh\frac{\pi}{a}v \Longleftrightarrow v = \frac{a}{\pi}\cosh^{-1}x;$$

and when $w = u$ $(0 < u < a)$, we have

$$x = -\cos\frac{\pi}{a}u \Longleftrightarrow u = \frac{a}{\pi}\cos^{-1}(-x).$$

First, we need to map the boundary conditions on w plane to z plane

$$f_1(v) = f_1(\frac{a}{\pi}\cosh^{-1}(-x)),$$

$$f_2(u) = f_2(\frac{a}{\pi}\cos^{-1}(-x)),$$

$$f_3(v) = f_3(\frac{a}{\pi}\cos^{-1}(-x)).$$

Apply the formula from Section 4.3.1 to find the solution to the z plane

$$\phi(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\tau)}{y^2 + (x - \tau^2)^2}\,\mathrm{d}\tau$$

$$= \frac{y}{\pi}\Big(\int_{-\infty}^{-1} \frac{f_1(\frac{a}{\pi}\cosh^{-1}(-\tau))}{y^2+(x-\tau^2)^2} + \int_{-1}^{1} \frac{f_2(\frac{a}{\pi}\cos^{-1}(-\tau))}{y^2+(x-\tau^2)^2} + \int_{1}^{\infty} \frac{f_3(\frac{a}{\pi}\cosh^{-1}\tau)}{y^2+(x-\tau^2}\Big)\,\mathrm{d}\tau \tag{35}$$

For $\tau < -1$, let $t = \frac{a}{\pi}\cosh^{-1}(-\tau)$, then $\tau = -\cosh\frac{\pi}{a}t$ and $\mathrm{d}\tau = -\frac{\pi}{a}\sinh\frac{\pi}{a}t\,\mathrm{d}t$;
for $-1 < \tau < 1$, let $t = \frac{a}{\pi}\cos^{-1}(-\tau)$, then $\tau = -\cos\frac{\pi}{a}t$ and $\mathrm{d}\tau = \frac{\pi}{a}\sin\frac{\pi}{a}t\,\mathrm{d}t$;
for $\tau > 1$, let $t = \frac{a}{\pi}\cosh^{-1}\tau$, then $\tau = \cosh\frac{\pi}{a}t$ and $\mathrm{d}\tau = \frac{\pi}{a}\sinh\frac{\pi}{a}t$.

Based on equaions 33 and 34, it follows from equation 35 that

$$\begin{aligned}
\phi(x(u,v),y(u,v)) = \frac{\sin\frac{\pi}{a}u\sinh\frac{\pi}{a}v}{\pi}\Big(\int_0^{\infty} &\frac{\frac{\pi}{a}\sinh(\frac{\pi}{a}t)f_1(t)}{\sin^2\frac{\pi}{a}u\sinh^2\frac{\pi}{a}v+(\cosh\frac{\pi}{a}t-\cos\frac{\pi}{a}u\cosh\frac{\pi}{a}v)^2}\,\mathrm{d}t \\
&+\int_0^{a} \frac{\frac{\pi}{a}\sin(\frac{\pi}{a}t)f_2(t)}{\sin^2\frac{\pi}{a}u\sinh^2\frac{\pi}{a}v+(\cos\frac{\pi}{a}t-\cos\frac{\pi}{a}u\cosh\frac{\pi}{a}v)^2}\,\mathrm{d}t \\
&+\int_0^{\infty} \frac{\frac{\pi}{a}\sinh(\frac{\pi}{a}t)f_3(t)}{\sin^2\frac{\pi}{a}u\sinh^2\frac{\pi}{a}v+(\cosh\frac{\pi}{a}t+\cos\frac{\pi}{a}u\cosh\frac{\pi}{a}v)^2}\,\mathrm{d}t\Big) \\
= \frac{\sin\frac{\pi}{a}u\sinh\frac{\pi}{a}v}{a}\Big(\int_0^{\infty} &\frac{\sinh(\frac{\pi}{a}t)f_1(t)}{\sin^2\frac{\pi}{a}u\sinh^2\frac{\pi}{a}v+(\cosh\frac{\pi}{a}t-\cos\frac{\pi}{a}u\cosh\frac{\pi}{a}v)^2}\,\mathrm{d}t \\
&+\int_0^{a} \frac{\sin(\frac{\pi}{a}t)f_2(t)}{\sin^2\frac{\pi}{a}u\sinh^2\frac{\pi}{a}v+(\cos\frac{\pi}{a}t-\cos\frac{\pi}{a}u\cosh\frac{\pi}{a}v)^2}\,\mathrm{d}t \\
&+\int_0^{\infty} \frac{\sinh(\frac{\pi}{a}t)f_3(t)}{\sin^2\frac{\pi}{a}u\sinh^2\frac{\pi}{a}v+(\cosh\frac{\pi}{a}t+\cos\frac{\pi}{a}u\cosh\frac{\pi}{a}v)^2}\,\mathrm{d}t\Big).
\end{aligned}$$

In conclusion, the solution to Laplace's equation on semi-infinite stripe is

$$\begin{aligned}
\phi(u,v) = \frac{\sin\frac{\pi}{a}u\sinh\frac{\pi}{a}v}{a}\Big(\int_0^{\infty} &\frac{\sinh(\frac{\pi}{a}t)f_1(t)}{\sin^2\frac{\pi}{a}u\sinh^2\frac{\pi}{a}v+(\cosh\frac{\pi}{a}t-\cos\frac{\pi}{a}u\cosh\frac{\pi}{a}v)^2}\,\mathrm{d}t \\
&+\int_0^{a} \frac{\sin(\frac{\pi}{a}t)f_2(t)}{\sin^2\frac{\pi}{a}u\sinh^2\frac{\pi}{a}v+(\cos\frac{\pi}{a}t-\cos\frac{\pi}{a}u\cosh\frac{\pi}{a}v)^2}\,\mathrm{d}t \\
&+\int_0^{\infty} \frac{\sinh(\frac{\pi}{a}t)f_3(t)}{\sin^2\frac{\pi}{a}u\sinh^2\frac{\pi}{a}v+(\cos\frac{\pi}{a}t+\cos\frac{\pi}{a}u\cosh\frac{\pi}{a}v)^2}\,\mathrm{d}t\Big).
\end{aligned}$$

□

If we compare the second integral of the above with the solution 16 in Section 3.4, we obtain the same result.

4.3.2 Modeling of fluid flow on a corner

Another example of solving Laplace's equation under conformal transform is on the first quadrant $\{(u,v)|u > 0, v > 0\}$. The boundary conditions are given on $u > 0$ and $v > 0$. The problem can be stated as

$$\begin{cases} \Phi_{uu} + \Phi_{vv} & = 0 \\ \Phi(u,0) & = f_1(u) \\ \Phi(0,v) & = f_2(v). \end{cases}$$

By the SC [7] mapping formula, we find the mapping function with coefficients undetermined.

$$w = f(z) = C \int z^{-\frac{1}{2}}$$

$$= Az^{\frac{1}{2}} + B,$$

where A, B are arbitrary coefficients, with $C = \frac{1}{2}A$

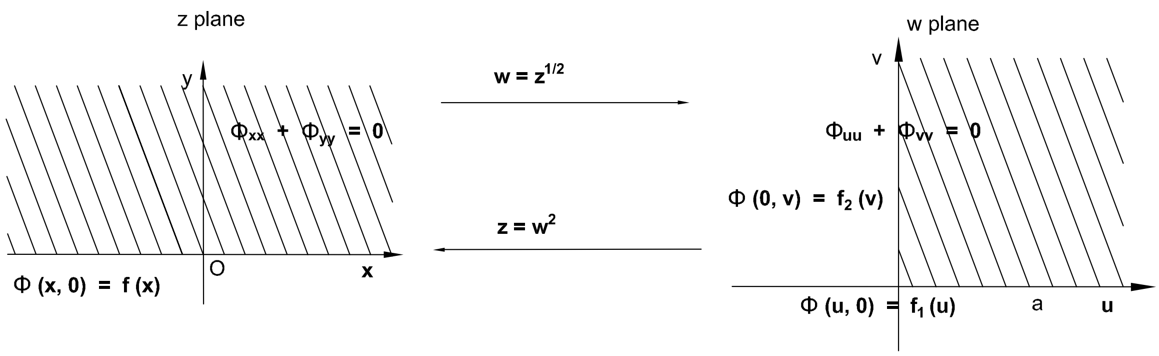

Figure 13: Mapping from the upper half plane to the top-right corner

For simplicity, we choose $f(0) = 0$ and set $A = 1$, and it gives us $w = \sqrt{z}$. The inverse is $z = w^2$.Expressed in Cartesian form, $z(x, y)$ and $w(u.v)$ are related by $x + iy = (u + iv)^2 = u^2 - v^2 + 2iuv$.

Along the positive u axis with $u > 0$ and $v = 0$, the relation is simplified as $u = \sqrt{x}$. In the same way, with $u = 0$ and $v > 0$, it yields $v = \sqrt{-x}$.

Hence, the boundary conditions $f(x)$ is found as the following

$$f(x) = \begin{cases} f_2(\sqrt{-x}) & x < 0, \\ f_1(\sqrt{x}) & x > 0. \end{cases}$$

[7]"SC" stands for Schwarz-Christoffel.

To avoid ambiguity in the integrating process, we shall write $f(\tau)$ in place of $f(x)$. For $\tau < 0$ let $t = \sqrt{-\tau}$, then it follows $\tau = -t^2$ and $\mathrm{d}\tau = -2t\,\mathrm{d}t$; and for $\tau > 0$ let $t = \sqrt{\tau}$, the it follows $\tau = t^2$ and $\mathrm{d}\tau = 2t\,\mathrm{d}t$.

Reusing the the formula of solutions we derived on upper-half plane, we have

$$\begin{aligned}\Phi(x,y) &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{y^2+(x-\tau)^2}\ \mathrm{d}\tau \\ &= \frac{2uv}{\pi}\left(\int_{-\infty}^{0} \frac{f_2(\sqrt{-\tau})}{(2uv)^2+(u^2-v^2-\tau)^2}\,\mathrm{d}\tau + \int_0^{\infty} \frac{f_1(\sqrt{\tau})}{(2uv)^2+(u^2-v^2-\tau)^2}\,\mathrm{d}\tau\right) \\ &= \frac{4uv}{\pi}\left(\int_0^{\infty} \frac{tf_2(t)}{(2uv)^2+(u^2-v^2+t^2)^2}\,\mathrm{d}t + \int_0^{\infty} \frac{tf_1(t)}{(2uv)^2+(u^2-v^2-t^2)^2}\,\mathrm{d}t\right).\end{aligned}$$

□

4.3.3 Modeling of fluid flow over a corner

By SC mapping formula, the conformal function is

$$\begin{aligned} w = f(z) = C\int z^{\frac{1}{2}} \\ = Az^{\frac{3}{2}} + B,\end{aligned}$$

where A, B are some constants with $C = \frac{3}{2}A$.

We choose principal branch, and we assume the correspondence $f(0) = 0$ and $f(-1) = -1$. Then, the mapping is function is

$$w = -i\,z^{\frac{3}{2}} = -i\,|z|^{\frac{3}{2}}\,e^{i\frac{3}{2}(\operatorname{Arg} z)},$$

with $0 < \operatorname{Arg} z < \pi$.

Since the mapping is one-to-one, we can find its inverse. If we choose the principal branch again on $w-$plane, then the the inverse function is

$$z = (i\,w)^{\frac{2}{3}} = |w|^{\frac{2}{3}}\,e^{i\frac{2}{3}\arg(iw)} = |w|^{\frac{2}{3}}\,e^{i\frac{2}{3}(\operatorname{Arg} w + \frac{\pi}{2})}, \tag{36}$$

with $-\frac{\pi}{2} < \operatorname{Arg} w < \pi$.

On the boundary $\{u < 0\} \cap \{v = 0\}$, we find $u = -(-x)^{\frac{3}{2}}$ as follows:

$$\begin{aligned} w = u = -i\,|z|^{\frac{3}{2}}\,e^{i\frac{3}{2}(\operatorname{Arg} z)} < 0 \implies -i\,e^{i\frac{3}{2}\operatorname{Arg} z} = -1 \implies -i\,e^{i\frac{3}{2}\operatorname{Arg} z} = -i \implies \operatorname{Arg} z = \pi \\ \implies z = x \text{ with } x < 0 \implies u = -(-x)^{\frac{3}{2}}.\end{aligned}$$

On the boundary $\{u = 0\} \cap \{v < 0\}$, we find $v = -x^{\frac{3}{2}}$ as follows:

$$w = i\,v = -i\,|z|^{\frac{3}{2}}\,e^{i\frac{3}{2}\operatorname{Arg} z} \implies v = -|z|^{\frac{3}{2}}\,e^{i\frac{3}{2}\operatorname{Arg} z} < 0 \implies e^{i\frac{3}{2}\operatorname{Arg} z} = 1 \implies \operatorname{Arg} z = 0$$

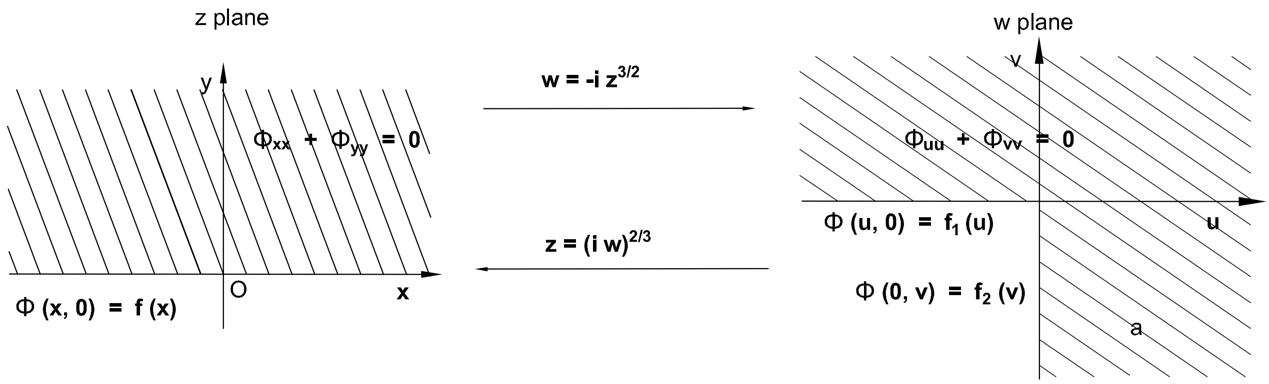

Figure 14: Mapping from upper half-plane to the plane excluding the third quadrant

$$\Longrightarrow z = x \text{ with } x > 0 \Longrightarrow v = -x^{\frac{3}{2}}.$$

Therefore, the boundary condition $f_1(u)$ and $f_2(v)$ can be parameterized as

$$f(x) = \begin{cases} f_1(u(x)) = f_1(-(-x)^{\frac{3}{2}}) & x < 0 \\ f_2(v(x)) = f_2(-x^{\frac{3}{2}}) & x > 0. \end{cases} \tag{37}$$

Replace x with τ in equation 37, we get

$$f(\tau) = \begin{cases} f_1(-(-\tau)^{\frac{3}{2}}) & \tau < 0 \\ f_2(-\tau^{\frac{3}{2}}) & \tau > 0. \end{cases}$$

Let $t = \begin{cases} -(-\tau)^{\frac{3}{2}}, \tau < 0 \\ -\tau^{\frac{3}{2}}, \tau > 0 \end{cases}$, then $\tau = \begin{cases} -(-t)^{\frac{2}{3}}, t \in (-\infty, 0) \\ (-t)^{\frac{2}{3}}, t \in (0, -\infty) \end{cases}$ and $d\tau = \begin{cases} \frac{2}{3}(-t)^{-\frac{1}{3}}\, dt, t \in (-\infty, 0) \\ -\frac{2}{3}(-t)^{-\frac{1}{3}}\, dt, t \in (0, -\infty) \end{cases}$.

The following is derived from equation 36

$$x = |w|^{\frac{2}{3}} \cos\frac{2}{3}(\operatorname{Arg} w + \frac{\pi}{2}) = (u^2 + v^2)^{\frac{1}{3}} \cos(\frac{2}{3}\operatorname{Arg} w + \frac{\pi}{3})$$

and

$$y = |w|^{\frac{2}{3}} \sin\frac{2}{3}(\operatorname{Arg} w + \frac{\pi}{2}) = (u^2 + v^2)^{\frac{1}{3}} \sin(\frac{2}{3}\operatorname{Arg} w + \frac{\pi}{3}).$$

With the above substitutions, we can derive the general formula for the solution

$$\Phi(u, v)$$

$$= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{y^2 + (x-\tau)^2}\, d\tau$$

$$= \frac{2(u^2+v^2)^{\frac{1}{3}} \cos(\frac{2}{3}\operatorname{Arg} w + \frac{\pi}{3})}{3\pi}$$

$$\int_{-\infty}^{0}\Bigg(\frac{(-t)^{-\frac{1}{3}}f_1(t)}{\Big((u^2+v^2)^{\frac{1}{3}}\cos(\frac{2}{3}\operatorname{Arg}w+\frac{\pi}{3})\Big)^2+\Big((u^2+v^2)^{\frac{1}{3}}\sin(\frac{2}{3}\operatorname{Arg}w+\frac{\pi}{3})+(-t)^{\frac{2}{3}}\Big)^2}$$

$$+\frac{(-t)^{-\frac{1}{3}}f_2(t)}{\Big((u^2+v^2)^{\frac{1}{3}}\cos(\frac{2}{3}\operatorname{Arg}w+\frac{\pi}{3})\Big)^2+\Big((u^2+v^2)^{\frac{1}{3}}\sin(\frac{2}{3}\operatorname{Arg}w+\frac{\pi}{3})-(-t)^{\frac{2}{3}}\Big)^2}\Bigg)\,\mathrm{d}t.$$

□

Appendix A Eigenvalue problems

Eigenvalue problems are a special type of homogeneous second-order ODE[8] with boundary conditions, and they have a set of solutions. In this article, Periodic boundary condition applies. Given the boundary value problem:

$$\begin{cases} u''(x) &= \pm\lambda u \\ u(-\pi) &= u(\pi) \\ u'(-\pi) &= u'(\pi), \end{cases}$$

where $\lambda \geqslant 0$.

Case A.1. $u'' = \lambda u$ $(\lambda \neq 0)$.

$$u = C_1 \cosh\sqrt{\lambda}x + C_2 \sinh\sqrt{\lambda}x$$

$$u' = \sqrt{\lambda}C_1 \sinh\sqrt{\lambda}x + \sqrt{\lambda}C_2 \cosh\sqrt{\lambda}x$$

$$u(-\pi) = u(\pi)$$

$$\Longrightarrow C_1 \cosh\sqrt{\lambda}\pi - C_2 \sinh\sqrt{\lambda}\pi = C_1 \cosh\sqrt{\lambda}\pi + C_2 \sinh\sqrt{\lambda}\pi$$

$$u'(-\pi) = u'(\pi)$$

$$\Longrightarrow -\sqrt{\lambda}C_1 \sinh\sqrt{\lambda}\pi + \sqrt{\lambda}C_2 \cosh\sqrt{\lambda}\pi = \sqrt{\lambda}C_1 \sinh\sqrt{\lambda}\pi + \sqrt{\lambda}C_2 \cosh\sqrt{\lambda}\pi$$

$$\therefore C_1 = 0, C_2 = 0.$$

$$\therefore \textit{There is no non-trivial solution.}$$

Case A.2. $\lambda = 0$.

$$u'' = 0 \Longrightarrow u = C_1 x + C_2$$

$$u(-\pi) = u(\pi) \Longrightarrow -C_1\pi + C_2 = C_1\pi + C_2$$

$$u'(-\pi) = u'(\pi) \Longrightarrow C_1 = C_1$$

$$\therefore C_1 = 0$$

[8]Ordinary differential equations

$$\therefore \textit{Eigenvalue is } 1\textit{, and the eigenfuction is } u = 1$$

Case A.3. $u'' = -\lambda u$ $(\lambda \neq 0)$.

$$u = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$$

$$u' = -\sqrt{\lambda}C_1 \sin\sqrt{\lambda} + \sqrt{\lambda}C_2 \cos\sqrt{\lambda}$$

$$u(-\pi) = u(\pi)$$

$$\Longrightarrow C_1 \cos\sqrt{\lambda}\pi - C_2 \sin\sqrt{\lambda}\pi = C_1 \cos\sqrt{\lambda}\pi + C_2 \sin\sqrt{\lambda}\pi$$

$$u'(-\pi) = u'(\pi)$$

$$\Longrightarrow \sqrt{\lambda}C_1 \sin\sqrt{\lambda}\pi + \sqrt{\lambda}C_2 \cos\sqrt{\lambda}\pi = -\sqrt{\lambda}C_1 \sin\sqrt{\lambda}\pi + \sqrt{\lambda}C_2 \cos\sqrt{\lambda}\pi$$

$$\therefore \sin\sqrt{\lambda}\pi = 0$$

Therefore, we have $\lambda = n^2$ *for* $n = 1, 2, 3\ldots$ *and* $u = a_n \cos nx + b_n \sin nx$.

As a result from three cases, we conclude

$$u(x) = \begin{cases} 1 & n = 0, \\ a_n \cos nx + b_n \sin nx & n = 1, 2, 3\ldots. \end{cases}$$

Appendix B Fourier Transform formula and conditions

If the conditions for Fourier transform are all satisfied, the following formula apply.

Fourier Transform:

$$\hat{u}(\omega) = \int_{-\infty}^{\infty} u(t)e^{-i\omega t}\,\mathrm{d}t$$

Inverse Transform:

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\omega)e^{i\omega t}\,\mathrm{d}\omega$$

Appendix C Proof of mapping between circles and lines

Four cases are considered: lines or circles through or not through the origin. Suppose $z = x + iy$ is the domain, and $w = u + iv$ is the image. The curves in the domain can be written in form of

$$Ax^2 + Ay^2 + Bx + Cy + D = 0 \tag{38}$$

with $\left(\frac{B}{2A}\right)^2 + \left(\frac{C}{2A}\right)^2 - D > 0$.

Relating z and w to find the image curves,

$$w = \frac{1}{z} \iff z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+V^2}$$

$$(38) \implies A\frac{u^2}{(u^2+v^2)^2}A\frac{(-v)^2}{(u^2+v^2)^2} + B\frac{u}{u^2+v^2} + C\frac{-v}{u^2+v^2} + D = 0$$

$$D(u^2+v^2) + Bu - Cv + A = 0 \tag{39}$$

Case C.1. *Line through origin ($A = 0$, $D = 0$).*

$$(39) \implies Bu - Cv = 0$$

∴ *Lines through origin are mapped to lines through origin.*

Case C.2. *Line not through origin ($A = 0$, $D \neq 0$).*

$$(39) \implies D(u^2+v^2) + Bu - Cv = 0$$

∴ *Lines not through origin are mapped to circles through origin.*

Case C.3. *Circle through origin ($A \neq 0$, $D = 0$).*

$$(39) \implies Bu - Cv + A = 0$$

∴ *Circles through origin are mapped to lines not through origin.*

Case C.4. *Circle not through origin ($A \neq 0$, $D \neq 0$).*

$$(39) \implies D(u^2+v^2) + Bu - Cv + A = 0$$

∴ *Circles not through origin are mapped to circles not through origin.*

References

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