The Lattice Method for Finding Conserved Quantities of Dynamical Systems

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Talk Overview

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Motivations

Dynamical systems are 1st order systems of ordinary differential equations (ODEs)

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$$\begin{aligned} \dot{x}(t) &= v(t) \\ \dot{v}(t) &= -\frac{F(t)}{m} \end{aligned}$$

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- **Disease modelling:** SIR model

$$\begin{aligned} \dot{S}(t) &= -\frac{\beta I(t)S(t)}{N} \\ \dot{I}(t) &= \frac{\beta I(t)S(t)}{N} - \gamma I(t) \\ \dot{R}(t) &= -\gamma I(t) \end{aligned}$$

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- **Machine learning:** Optimization via gradient descent

$$\dot{\boldsymbol{x}}(t) = -\nabla f(\boldsymbol{x}(t))$$

Motivation continued

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In our research, we focus on the problem of finding conserved quantities for polynomial dynamical systems.

Talk Overview

Dynamical system

Definition

A continuous dynamical system is an ODE system in $\mathbb{R}^N$ of the form

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(t, \boldsymbol{x}(t)) \\ \boldsymbol{x}(t_0) &= \boldsymbol{x}_0, \end{aligned} \tag{1}$$

where $\boldsymbol{x}(t) = [x_1(t), \ldots, x_N(t)]^T$ is the unknown vector-valued function, $\boldsymbol{f}(t, \boldsymbol{x}) = [f_1(t, \boldsymbol{x}), \ldots, f_N(t, \boldsymbol{x})]^T$ is the RHS of the ODE (1), and $t \in I \subset \mathbb{R}$, $(t_0, \boldsymbol{x}_0) \in I \times \mathbb{R}^N$.

Some Facts

ODE (1) has a unique solution locally if $\boldsymbol{f}$ is continuous in t and Lipschitz continuous in $\boldsymbol{x}$, near $(t_0, \boldsymbol{x}_0)$.

Definition

We say that (1) is a **polynomial system** if each component f_i is a polynomial in $x_1, \ldots, x_N$.

Examples of Dynamical Systems

Hamiltonian system

A special class of system called **Hamiltonian system** in $\mathbb{R}^{2N}$ has the form

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{y}} \end{bmatrix} = \begin{bmatrix} \nabla_{\boldsymbol{y}} \mathcal{H}(\boldsymbol{x}, \boldsymbol{y}) \\ -\nabla_{\boldsymbol{x}} \mathcal{H}(\boldsymbol{x}, \boldsymbol{y}) \end{bmatrix}, \qquad \text{(H.S.)}$$

where $\mathcal{H}(\boldsymbol{x}, \boldsymbol{y}) : \mathbb{R}^{2N} \to \mathbb{R}$ is the Hamiltonian.

Example: The energy function $\mathcal{H}(x, y) = \dfrac{y^2}{2m} + V(x)$ can be expressed as dynamcal system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{y}{m} \\ -\frac{\partial V}{\partial x} \end{bmatrix}.$$

- Recall from physics this is called a "conservative system".

Examples of Dynamical Systems Continued

Quadratic 2D system

A general quadratic 2D system is of the from

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} Ax + By + Cx^2 + Dxy + Ey^2 \\ Fx + Gy + Hx^2 + Ixy + Jy^2 \end{bmatrix}, \tag{Q2D}$$

where $A, B, \ldots, J$ are real constants.

Example: Predator-prey model

$$\begin{aligned} \dot{x} &= x(1-y) \\ \dot{y} &= y(1-x) \end{aligned}. \tag{P.P.}$$

- This is a model for population dynamics between two species.

Examples of Dynamical Systems Continued

Cubic 2D system

A general cubic 2D system is given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} Ax + By + Cx^2 + Dxy + Ey^2 + Fx^3 + Gx^2y + Hxy^2 + Iy^3 \\ Jx + Ky + Lx^2 + Mxy + Ny^2 + Ox^3 + Px^2y + Qxy^2 + Ry^3 \end{bmatrix}, \tag{C2D}$$

where $A, B, \ldots, R$ are constants.

Example: Van der Pol Oscillator

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -x + \mu(1 - x^2)y \end{bmatrix}, \tag{V.D.P.}$$

where $\mu > 0$ is constant.

- This is a type of nonlinear oscillator.

Examples of Dynamical Systems Continued

Quadratic 3D system

A general quadratic 3D system is of the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_1x + B_1y + C_1z + D_1x^2 + E_1y^2 + F_1z^2 + G_1xy + H_1yz + I_1xz \\ A_2x + B_2y + C_2z + D_2x^2 + E_2y^2 + F_2z^2 + G_2xy + H_2yz + I_2xz \\ A_3x + B_3y + C_3z + D_3x^2 + E_3y^2 + F_3z^2 + G_3xy + H_3yz + I_3xz, \end{bmatrix} \tag{Q3D}$$

where $A_i, \ldots, I_i,\ i = 1, 2, 3,$ are constants.

Example: Lorenz system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y - x) \\ x(\rho - z) - y \\ xy - \beta z \end{bmatrix}, \tag{L.S.}$$

where σ, ρ, β are constants.

- This is a well-known quadratic 3D example which exhibits "chaotic behavior" for certain range of parameters.

Conserved quantities of Dynamical Systems

Definition

A **conserved quantity (CQ)** of (1) is a scalar function $\Psi(t, \boldsymbol{x}) : I \times \mathbb{R}^N \to \mathbb{R}$ such that $\frac{\mathrm{d}}{\mathrm{d}\,t}\Psi(t, \boldsymbol{x}) = 0$ whenever $\boldsymbol{x}(t)$ is a solution of (1).
We say that a conserved quantity is **time-independent** if Ψ does not depend on t explicitly, otherwise, it is said to be **time-dependent**.

Example: Hamiltonian system

For the Hamiltonian system (H.S.), the CQ is the function $\mathcal{H}(\boldsymbol{x}, \boldsymbol{y})$, which is called the energy function.

- Note that it is time-independent, which means energy is preserved on solution curves.

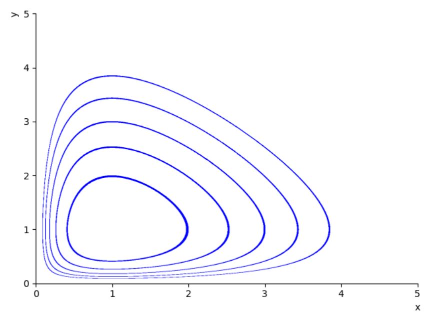
$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Psi = \nabla_{\boldsymbol{x}} H\dot{\boldsymbol{x}} + \nabla_{\boldsymbol{y}} H\dot{\boldsymbol{y}} = \nabla_{\boldsymbol{x}} H\nabla_{\boldsymbol{y}} H - \nabla_{\boldsymbol{y}} H\nabla_{\boldsymbol{x}} H = 0.$$

Conserved quantities of Dynamical Systems Continued

Example: Predator-Prey Model

Lotka–Volterra [1920s] discovered a CQ for (P.P.) system

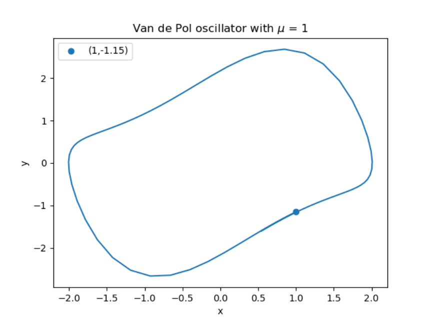
$$\Psi(x, y) = x - \log x + y - \log y$$



- The solution always stays on the same level set determined by their I.C..
- If the level set is compact, then the solution exists for all $t \in \mathbb{R}$.

Conserved quantities of Dynamical Systems Continued

Example: Van der Pol Oscillator



- Van der pol oscillator is a case of (C2D) without known CQs.
- However, there are "limit cycles", as shown left.
- Time-dependent CQ can imply a limit cycle for a zero set of f.

Suppose there exists a CQ of the form $\Psi(t, x, \mu) := e^{\lambda t} f(x, y)$ with $\lambda > 0$.

$$\text{CQ} \implies e^{\lambda t} f(x(t), y(t)) = C \implies f(x(t), y(t)) = Ce^{-\lambda t}$$

$$\implies \lim_{t \to \infty} f(x(t), y(t)) = 0, \text{ie, zero set of } f \text{ describes the limit cycle.}$$

Question: Does a time-dependent conserved quantity of this kind exist?

Conserved quantities of Dynamical Systems Continued

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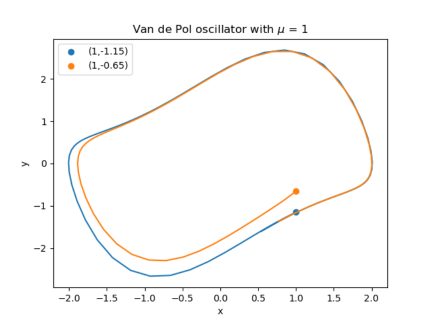

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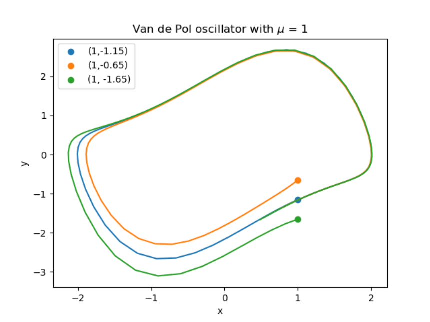

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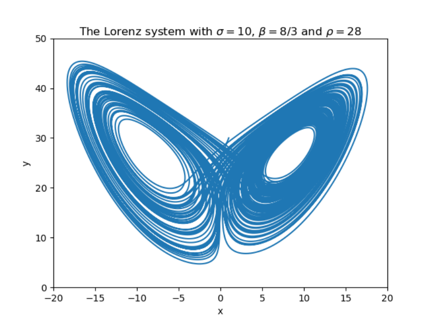
Conserved quantities of Dynamical Systems Continued

Example: Lorenz system



- The Lorenz system is a classical example of chaotic system for certain range of parameters.

- So far there are six known conserved quantities [AS81][Kus83].
- However, all of theses conserved quantities exist in non-chaotic regime of parameters

$f(x,y,z)$	cofactor λ	parameters
$x^2 - 2\sigma z$	-2σ	$\beta = 2\sigma$
$-\rho x^2 + \frac{1}{3}y^2 + \frac{2}{3}xy + x^2 z - \frac{3}{4}x^4$	$-\frac{4}{3}$	$\beta = 0, \sigma = \frac{1}{3}$
$y^2 + z^2$	-2	$\beta = 1, \rho = 0$
$4(1-\rho)z + \rho x^2 + y^2 - 2xy + x^2 z - \frac{1}{4}x^4 =$	-4	$\beta = 4, \sigma = 1$
$-\rho x^2 + y^2 + z^2$	-2	$\beta = 1, \sigma = 1$
$\frac{1}{\sigma}(2\sigma - 1)^2 x^2 + \sigma y^2 - (4\sigma - 2)xy + x^2 z - \frac{1}{4s}x^4$	-4σ	$\beta = 6\sigma - 2, \rho = 2\sigma - 1$

Question: Do other time-dependent CQ exist in the chaotic regime?

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Darboux First Integral

We will focus on finding conserved quantities for polynomial systems in 2D, specifically of the form $\Psi(t,x,y) = e^{\lambda t} f(x,y)$, where $f(x,y)$ is polynomial.

Definition

Darboux first integral Ψ is a CQ of the form

$$\Psi(t,\boldsymbol{x}) = e^{\lambda t} f(\boldsymbol{x}),$$

where $f(\boldsymbol{x})$ is polynomial.

Examples:

- (Damped harmonic oscillator)[WBN17]
  $\Psi(x,y) = \frac{1}{2} e^{-\frac{\gamma t}{m}}(my^2 + \gamma xy + kx^2)$, if $\gamma = 0$ and $\Psi = H$.
- (Lorenz system) $\Psi(x,y,z) = e^{-2\sigma}(x^2 - 2\sigma z)$ if $\beta = 2\sigma$.
- We will see more later.

Lattice Method

For simplicity, we start with the quadratic 2D system (Q2D).

- Assume CQ of the form $\Psi(t, x, y) = e^{-\lambda} f(x, y)$, where $f(x, y) = \sum_{m,n=0}^{\infty} C_{m,n} x^m y^n$. So $f(x, y)$ includes polynomial of any degree.
- Then we have $0 = \dfrac{d\Psi}{dt} = e^{-\lambda t}(\dfrac{df}{dt} - \lambda f)$ by the chain rule.

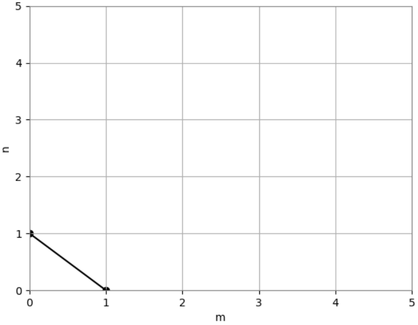
$$\implies f_x \dot{x} + f_y \dot{y} - \lambda f = 0$$

$$\begin{aligned}(\text{Q2D}) \implies & \sum_{m,n=0}^{\infty} (m+1) C_{m+1,n} x^m y^n (Ax + By + Cx^2 + Dxy + Ey^2) \\ & + \sum_{m,n=0}^{\infty} (n+1) C_{m,n+1} x^m y^n (Fx + Gy + Hx^2 + Ixy + Jy^2) \\ & -\lambda \sum_{m,n=0}^{\infty} C_{m,n} x^m y^n = 0 \end{aligned} \tag{2}$$

Lattice Method

In the next step, we simplify (2) by separating terms that have a factor of $x^l y^m$ for some $l, m \geq 2$ and grouping those that do not. we will have the following relations:

$$\text{Q2D-1a: } (A-\lambda)C_{1,0} + FC_{0,1} = 0$$
$$\text{Q2D-1b: } BC_{1,0} + (G-k)C_{0,1} = 0$$



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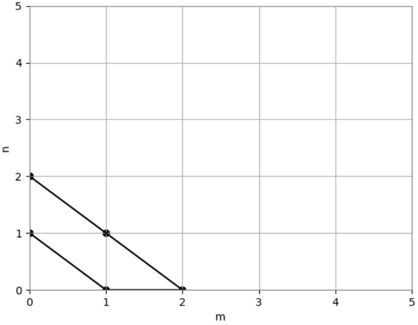
In the next step, we simplify (2) by separating terms that have a factor of $x^l y^m$ for some $l, m \geq 2$ and grouping those that do not. we will have the following relations:

Q2D-2: $2BC_{2,0} + (G + A - k)C_{1,1} + DC_{1,0} + IC_{0,1} + 2FC_{0,2} = 0$



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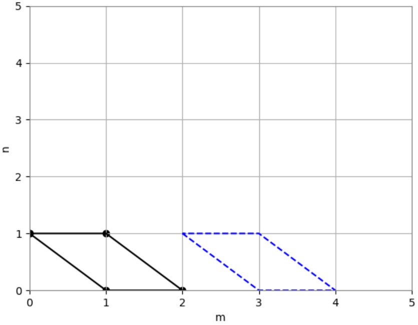
In the next step, we simplify (2) by separating terms that have a factor of $x^l y^m$ for some $l, m \geq 2$ and grouping those that do not. we will have the following relations:

Q2D-3:

$$(Am-k)C_{m,0} + FC_{m-1,1} + C(m-1)C_{m-1,0} + HC_{m-2,1} = 0, \; m \geq 2$$



Lattice Method

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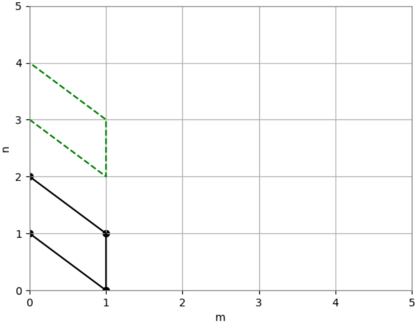
Q3D-4:

$$BC_{1,n-1} + (Gn-k)C_{0,n} + EC_{1,n-2} + J(n-1)C_{0,n-1} = 0,\ n \geq 2$$



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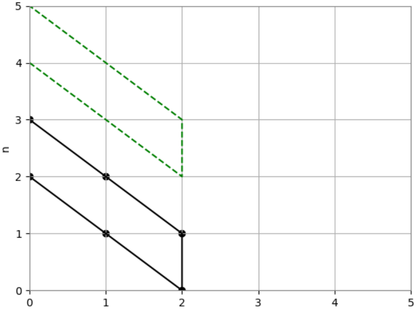
Q2D-5:

$$(A + Gn - k)C_{1,n} + F(n+1)C_{0,n+1} + 2BC_{2,n-1}$$

$$+[D + I(n-1)]C_{1,n-1} + InC_{0,n} + 2EC_{2,n-2} = 0, \; n \geq 2$$

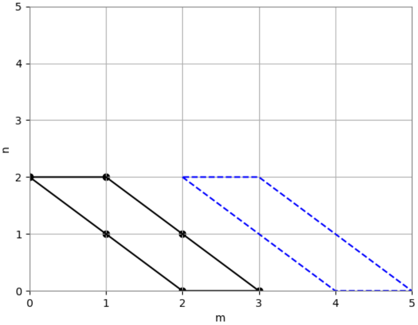


Lattice Method

In the next step, we simplify (2) by separating terms that have a factor of $x^l y^m$ for some $l, m \geq 2$ and grouping those that do not. we will have the following relations:

Q2D-6:

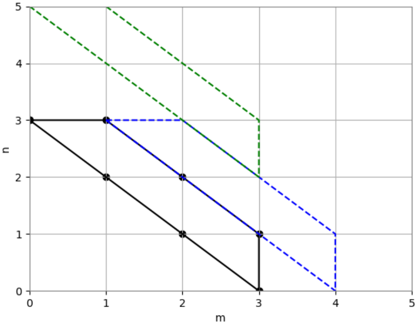
$$(Am + G - k)C_{m,1} + 2FC_{m-1,2} + B(m+1)C_{m+1,0} + [C(m-1) + l]C_{m-1,1}\, 2HC_{m-2,2} + DmC_{m,0} = 0,\ m \geq 2$$



Lattice Method

In the next step, we simplify (2) by separating terms that have a factor of $x^l y^m$ for some $l, m \geq 2$ and grouping those that do not. we will have the following relations:

$$\begin{aligned}\text{Q2D-7: } &(Am + Gn - k)C_{m,n} + F(n+1)C_{m-1,n+1} + B(m+1)C_{m+1,n-1} \\ &+[C(m-1) + In]C_{m-1,n} + H(n+1)C_{m-2,n+1} \\ &+[Dm + J(n-1)]C_{m,n-1} + E(m+1)C_{m+1,n-2} = 0,\ m, n \geq 2.\end{aligned}$$



Lattice Method: Matrix form for Q2D

Let M denote the sum of powers of x and y.

$M = 1$: $\begin{bmatrix} A-k & F \\ B & G-k \end{bmatrix} \begin{bmatrix} C_{1,0} \\ C_{0,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

$M = 2$: $\begin{bmatrix} 2A-k & F & 0 \\ 2B & G+A-k & 2F \\ 0 & B & 2G-k \end{bmatrix} \begin{bmatrix} C_{2,0} \\ C_{1,1} \\ C_{0,2} \end{bmatrix} = \begin{bmatrix} -C & -H \\ -D & -I \\ -E & -J \end{bmatrix} \begin{bmatrix} C_{1,0} \\ C_{0,1} \end{bmatrix}.$

$M = 3$: $\begin{bmatrix} 3A-k & F & 0 & 0 \\ 3B & G+2A-k & 2F & 0 \\ 0 & 2B & 2G+A-k & 3F \\ 0 & 0 & B & 3G-k \end{bmatrix} \begin{bmatrix} C_{3,0} \\ C_{2,1} \\ C_{1,2} \\ C_{0,3} \end{bmatrix} =$

$\begin{bmatrix} -2C & -H & 0 \\ -2D & -(C+I) & -2H \\ -2E & -(D+J) & -2I \\ -E & 0 & -2J \end{bmatrix} \begin{bmatrix} C_{2,0} \\ C_{1,1} \\ C_{0,2} \end{bmatrix}.$

$\vdots$

M: $(P_M - kI)\boldsymbol{v}_M = Q_M \boldsymbol{v}_{M-1}$, where

Lattice Method: Matrix form for Q2D

Define $\boldsymbol{v}_M = \begin{bmatrix} C_{M,0} & C_{M-1,1} & \dots & C_{1,M-1} & C_{0,M} \end{bmatrix}^T$, $P_M =$

$$\begin{bmatrix}
MA & F & 0 & \dots & 0 & & \\
MB & (M-1)A+G & 2F & 0 & \dots & 0 & \\
0 & (M-1)B & (M-2)A+2G & 3F & 0 & \dots & 0 \\
\ddots & \ddots & \ddots & & & & \\
0 & \dots & 0 & 3B & 2A+(M-2)G & (M-1)F & 0 \\
0 & \dots & 0 & 0 & 2B & A+(M-1)G & MF \\
0 & \dots & 0 & 0 & 0 & B & MG
\end{bmatrix},$$

and $Q_M =$

$$\begin{bmatrix}
C & H & 0 & \dots & 0 & & \\
D & (M-2)C+I & 2H & 0 & \dots & 0 & \\
(M-1)E & (M-2)D+J & (M-3)C+2I & 3H & & & \\
\ddots & \ddots & & & & & \\
0 & \dots & 0 & 3E & 2D+(M-3)J & C+(M-2)I & (M-1)H \\
0 & \dots & 0 & 0 & 2E & D+(M-2)J & (M-1)I \\
0 & \dots & 0 & 0 & 0 & E & (M-1)J
\end{bmatrix}$$

Lattice Method: Matrix equations for Q2D

We then obtain a sequence of matrix equations for (Q2D):

$$\begin{aligned}
(P_1 - \lambda I)\boldsymbol{v}_1 &= 0 \\
(P_2 - \lambda I)\boldsymbol{v}_2 + Q_1\boldsymbol{v}_1 &= 0 \\
&\vdots \\
(P_{n-1} - \lambda I)\boldsymbol{v}_{n-1} + Q_{n-2}\boldsymbol{v}_{n-2} &= 0 \\
(P_n - \lambda I)\boldsymbol{v}_n + Q_{n-1}\boldsymbol{v}_{n-1} &= 0 \\
Q_n\boldsymbol{v}_n &= 0.
\end{aligned}$$

Lattice Method: Matrix form for C2D

$s = 1$: $(P_1 - \lambda I)\boldsymbol{v}_1 = 0 \iff$

$$\begin{bmatrix} A-\lambda & J \\ B & K-\lambda \end{bmatrix} \begin{bmatrix} C_{1,0} \\ C_{0,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$s = 2$: $(P_2 - \lambda I)\boldsymbol{v}_2 + Q_1\boldsymbol{v}_1 = 0 \iff$

$$\begin{bmatrix} 2A-\lambda & J & 0 \\ 2B & A+K-\lambda & 2J \\ 0 & B & 2K-\lambda \end{bmatrix} \begin{bmatrix} C_{2,0} \\ C_{1,1} \\ C_{0,2} \end{bmatrix} + \begin{bmatrix} C & L \\ D & M \\ E & N \end{bmatrix} \begin{bmatrix} C_{1,0} \\ C_{0,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$s = 3$: $(P_3 - \lambda I)\boldsymbol{v}_3 + Q_2\boldsymbol{v}_2 + R_1\boldsymbol{v}_1 = 0 \iff$

$$\begin{bmatrix} 3A-\lambda & J & 0 & 0 \\ 3B & 2A+K-\lambda & 2J & 0 \\ 0 & 2B & A+2K-\lambda & 3J \\ 0 & 0 & B & 3K-\lambda \end{bmatrix} \begin{bmatrix} C_{3,0} \\ C_{2,1} \\ C_{1,2} \\ C_{0,3} \end{bmatrix}$$

$$+ \begin{bmatrix} 2C & L & 0 \\ 2D & C+M & 2L \\ 2E & D+N & 2M \\ 0 & E & 2N \end{bmatrix} \begin{bmatrix} C_{2,0} \\ C_{1,1} \\ C_{0,2} \end{bmatrix} + \begin{bmatrix} F & O \\ G & P \\ H & Q \\ I & R \end{bmatrix} \begin{bmatrix} C_{1,0} \\ C_{0,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$s=n$: $(P_n-\lambda I)\mathbf{v}_n+Q_{n-1}\mathbf{v}_{n-1}+R_{n-2}\mathbf{v}_{n-2}=0$, where

$$P_n=\begin{bmatrix} nA & J & 0 & \dots & & \\ nB & (n-1)A+K & 2J & 0 & \dots & \\ 0 & (n-1)A & (n-2)A+2K & 3J & 0 & \dots \\ & \ddots & \ddots & \ddots & & \\ \dots & \dots & 0 & 2B & A+(n-1)K & nJ \\ \dots & \dots & \dots & 0 & B & nK \end{bmatrix},$$

$$Q_n=$$

$$\begin{bmatrix} (n-1)C & L & 0 & \dots & & \\ (n-1)D & (n-2)C+M & 2L & 0 & \dots & \\ (n-1)E & (n-2)D+N & (n-3)C+2M & 3L & 0 & \dots \\ 0 & (n-2)E & (n-3)D+2N & (n-4)C+3M & 4L & \\ & \ddots & \ddots & \ddots & & \\ 0 & \dots & & 2E & D+(n-2)N & (n-1)M \\ 0 & \dots & & & E & (n-1)N \end{bmatrix}$$

and $$R_n=\begin{bmatrix} (n-2)F & O & \dots & & & \\ (n-2)G & (n-3)F+P & 2O & & & \\ (n-2)H & (n-3)G+Q & (n-4)F+2P & 3O & & \\ (n-2)I & (n-3)H+R & (n-4)G+2Q & (n-5)F+3P & 4O & \\ & \ddots & \ddots & \ddots & & \\ 0 & \dots & & & & (n-2)O \\ 0 & \dots & & & & (n-2)P \\ 0 & \dots & & & & (n-2)Q \\ 0 & \dots & & & I & (n-2)R \end{bmatrix}$$

Lattice Method: Matrix equations for C2D

Similar to Q2D, we obtain a sequence of matrix equations (C2D):

$$
\begin{aligned}
(P_1 - \lambda I)\boldsymbol{v}_1 &= 0 \\
(P_2 - \lambda I)\boldsymbol{v}_2 + Q_1\boldsymbol{v}_1 &= 0 \\
(P_3 - \lambda I)\boldsymbol{v}_3 + Q_2\boldsymbol{v}_2 + R_1\boldsymbol{v}_1 &= 0 \\
&\vdots \\
(P_n - \lambda I)\boldsymbol{v}_n + Q_{n-1}\boldsymbol{v}_{n-1} + R_{n-2}\boldsymbol{v}_{n-2} &= 0 \\
Q_n\boldsymbol{v}_n + R_{n-1}\boldsymbol{v}_{n-1} &= 0 \\
R_n\boldsymbol{v}_n &= 0.
\end{aligned}
$$

- Note that C2D has one more term R_n than Q2D.

Algorithm for Q2D and C2D: Main idea

Since Q2D is a special case of C2D, we only consider C2D.

- Our goal is to determine those coefficient vectors $\boldsymbol{v}_k$ for f.
- For a polynomial f, only finite many $\boldsymbol{v}_k$'s are nonzero.
- Say we are interested in polynomial f with lowest order term $\boldsymbol{v}_i$ and highest order term $\boldsymbol{v}_n$.
- Thus, let us assume that i is the smallest index such that $\boldsymbol{v}_i \neq 0$ and n is the smallest index such that $\boldsymbol{v}_k = 0$ for all $k > n$.
- Then it requires $(P_i - \lambda I)\boldsymbol{v}_i = 0$, ie, λ and $\boldsymbol{v}_i$ are eigenvalue and eigenvector of P_i. So the staring point is to find the eigenvalues and corrsponding eigenvectors for P_i for given i.
- From then on, we sequentially solve $(P_k - \lambda I)\boldsymbol{v}_k + Q_{k-1}\boldsymbol{v}_{k-1} + R_{k-2}\boldsymbol{v}_{k-2} = 0$ for $\boldsymbol{v}_k$, $k = i+1, \ldots, n$.
- Lastly, it also requires two terminal conditions $Q_n\boldsymbol{v}_n + R_{n-1}\boldsymbol{v}_{n-1} = 0$ and $R_n\boldsymbol{v}_n = 0$. If both satsfied, we have determined coefficients of f.

Algorithm for Q2D and C2D: Pseudocode

Pseudocode: Finding C.Q.s for the Cubic 2D case of degree n

function findCQ-C2D($P_1, P_2, \ldots,\ Q_1, Q_2, \ldots,\ R_1, R_2, \ldots$)

    // P, Q, R are the coefficient matrices.

    for $i = 1$ to n **do**

        for each eigenvalue λ and eigenvector $\boldsymbol{v}_i$ **do**

            Solve $(P_{i+1} - \lambda I)\boldsymbol{v}_{i+1} + Q_i \boldsymbol{v}_i = 0$ for $\boldsymbol{v}_{i+1}$

            for $j = i + 2$ to n **do**

                Solve $(P_j - \lambda I)\boldsymbol{v}_j + Q_{j-1}\boldsymbol{v}_{j-1} + R_{j-2}\boldsymbol{v}_{j-2} = 0$ for $\boldsymbol{v}_j$

            Compute $\boldsymbol{x}_1 = Q_n \boldsymbol{v}_n + R_{n-1}\boldsymbol{v}_{n-1}$ and $\boldsymbol{x}_2 = R_n \boldsymbol{v}_n$

            if $\boldsymbol{x}_1 = 0$ and $\boldsymbol{x}_2 = 0$ **then**

                A f is found with coefficients $\boldsymbol{v}_i, \ldots, \boldsymbol{v}_n$.

    return all such f

As we can see from above, regardless of complexity to find eigenvalues and eigenvectors, the time complexity is about $\mathcal{O}(n^3)$.

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Hamiltonian System

Consider Hamiltonian system

$$\dot{x} = Ax + By + Cx^2 - 2Jxy + Ey^2,$$
$$\dot{y} = Fx - Ay + Hx^2 - 2Cxy + Jy^2.$$

We let $\boldsymbol{v}_1 = 0$, and solve the following equations:

$(P_2 - \lambda I)\boldsymbol{v}_2 = 0 \iff$

$$\begin{bmatrix} 2A-\lambda & F & 0 \\ 2B & -\lambda & 2F \\ 0 & B & -2A-\lambda \end{bmatrix} \boldsymbol{v}_2 = 0 \implies \boldsymbol{v}_2 = \begin{bmatrix} -F & 2A & B \end{bmatrix}^T \text{ with } \lambda = 0.$$

$P_3\boldsymbol{v}_3 + Q_2\boldsymbol{v}_2 = 0 \iff$

$$\begin{bmatrix} 3A & F & 0 & 0 \\ 3B & A & 2F & 0 \\ 0 & 2B & -A & 3F \\ 0 & 0 & B & -3A \end{bmatrix} \boldsymbol{v}_3 = \begin{bmatrix} -2C & -H & 0 \\ 4J & C & -2H \\ -2E & J & 4C \\ 0 & -E & -2J \end{bmatrix} \begin{bmatrix} -F \\ 2A \\ B \end{bmatrix}$$

$$\implies \boldsymbol{v}_3 = \begin{bmatrix} -\frac{2}{3}H & 2C & -2J & \frac{2}{3}E \end{bmatrix}^T.$$

Hamiltonian System continued

$$Q_3 \boldsymbol{v}_3 = \begin{bmatrix} 3C & H & 0 & 0 \\ -6J & 0 & 2H & 0 \\ 3E & -3J & -3C & 3H \\ 0 & 2E & 0 & -6C \\ 0 & 0 & E & 3J \end{bmatrix} \begin{bmatrix} -\frac{2}{3}H \\ 2C \\ -2J \\ \frac{2}{3}E \end{bmatrix} = 0.$$

Thus, we find $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ with $\lambda = 0$, which gives
$\mathcal{H}(x, y) = -Fx^2 + 2Axy + By^2 - \frac{2}{3}Hx^3 + 2Cx^2y - 2Jxy^2 + \frac{2}{3}Ey^2$.

Non-Hamiltonian System

An non-Hamiltonian example is

$$\begin{aligned}\dot{x} &= x + x^2 + xy + x^3 + x^2y + xy^2\\ \dot{y} &= 2x + 2y + 2x^2 + xy + y^2 - x^2y - xy^2 - y^3.\end{aligned}$$

$$(P_1 - \lambda I)\boldsymbol{v}_1 = 0 \iff \begin{bmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix} \boldsymbol{v}_1 = 0$$

$\implies \boldsymbol{v}_1 = \begin{bmatrix} -2 & 0 \end{bmatrix}^T$ with $\lambda = 1$.

$(P_2 - \lambda I)\boldsymbol{v}_2 + Q_1 \boldsymbol{v}_1 = 0$

$$\iff \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \boldsymbol{v}_2 = - \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$\implies \boldsymbol{v}_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$.

We also check that

$$Q_2 \boldsymbol{v}_2 + R_1 \boldsymbol{v}_1 = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $R_2 \boldsymbol{v}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

Hence, the conserved quantity is given by

$$\Psi(x, y, t) = e^{-t}(-2x + xy).$$

Lorenz System

- Similarly, we also derived the lattice equations for the Lorenz system. We omit the general C3D case due to their complexity.
- We were also able to recover six known CQs for Lorenz system by solving a sequence of matrix equations.

Drawbacks on the algorithm

However, there are still some potential issues with our algorithm that we hope to improve on:

- At implementation level, it is hard to solve the problem $A\boldsymbol{x} = \boldsymbol{b}$, where A is a rank deficiency matrix but $\boldsymbol{b}$ is in the range of A, ie, a solution $\boldsymbol{x}$ does exist.
- Even if we solve the above problem, there are likely many classes of choices for solution $\boldsymbol{x}$, and so our algorithm has to search every path that might lead to a CQ. If later we run into the same scenario at higher dimension, it would have "domino effect".
- Our algorithm does not address complex eigenvalues.
- The time complexity is high for search of high-order polynomial CQ.

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Summary

- Introduced the lattice method to find Darboux First Intergals for Q2D and C2D case.
- Presented a general algorithm to find all possible Darboux First Intergals of degree up to *n*.
- Applied to non-trivial examples, as well as some special systems such as the Lorenz system.

Future work:

- Generalize to system with arbitrary polynomial order
- Look for other types of C.Q.s

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